

**PRACTICE MIDTERM #1**

**Question 1**

1. (2) $S_1$, $S_2$ are the null spaces of the maps $T : V \rightarrow \mathbb{R}$ given by $T(f) = f(a)$ where $a = 0, 1$ respectively. As the rank of this map is 1 (the constant polynomials map onto $\mathbb{R}$, for example), and the dimension of $V$ is 4, by the Rank-Nullity Theorem, the dimension of $S_1$ and $S_2$ is three.

2. To extend this linearly independent set to a basis, it suffices to add any vector outside the vector space $S_1 \cup S_2$, and hence the sum of their squares can only be zero when both are zero.

3. $S_3$ is not a subspace as it does not contain 0, for example.

4. $S_4$ is a subspace, being the intersection of the subspaces $S_1$ and $S_2$. It is of dimension two: $\{x(x - 1), x^2(x - 1)\}$ is a basis.

5. $S_5$ is not a subspace, since it is the union of two subspaces ($S_5 = S_1 \cup S_2$) and neither contains the other (to refer to an exercise many of you have seen in recitation). Directly, we have $x \in S_5$ and $x - 1 \in S_5$ but $f(x) = x + (x - 1) = 2x - 1 \notin S_5$, as $f(0) = -1$ and $f(1) = 1$.

6. $S_6$ is a subspace, as it is the null space of the linear map $T : V \rightarrow V$ given by $T(f) = f(0) + f(1)$ (you should check that this is a linear map).

7. $S_7$ is a subspace, since the equation $p(0)^2 + p(1)^2 = 0$ is equivalent to $p(0) = 0$ and $p(1) = 0$ (so in fact, $S_7 = S_1$). Since $p(0)$ and $p(1)$ are real numbers, their squares are nonnegative, and hence the sum of their squares can only be zero when both are zero.

**Question 2**

1. The subset $\{(0, 1, 3), (1, 2, 3), (2, 3, 1)\} = \{v_1, v_2, v_3\}$ is a basis. To prove this, first notice by inspection that $v_1$ and $v_2$ are not scalar multiples of each other, and are nonzero, so $\{v_1, v_2\}$ is a linearly independent set. Now, in order to show that we may add $v_3$ to this set without losing linear independence, it is sufficient to show that $v_3 \notin \text{span}(\{v_1, v_2\})$.

   Suppose by contradiction that we have $v_3 = av_1 + bv_2$, i.e. $v_3 \in \text{span}(\{v_1, v_2\})$. Then equating coordinates, we have $2 = b, 3 = a + 2b$, and $1 = 3a + 3b$. Substituting $b = 2$ in the second equation yields $3 = a + 4 \Rightarrow a = -1$, and plugging both of these into the third equation yields $1 = 3(-1) + 3(2) = 3$, a contradiction in the field $\mathbb{R}$. Therefore the set $\{v_1, v_2, v_3\}$ is linearly independent. As it size three, and we know the dimension of $\mathbb{R}^3$ is 3, it is a basis.

   To express $(3, 3, 3)$ in terms of this basis, we solve the equations $a(0, 1, 3) + b(1, 2, 3) + c(2, 3, 1) = (3, 3, 3)$, i.e. $b + 2c = 3, a + 2b + 3c = 3, 3a + 3b + c = 3$. Subtracting the second from the first, we get $a + b + c = 0$. Subtracting three times this from the third equation, we find $-2c = 3$, so $c = -3/2$. The original first equation $b + 2c = 3$ now gives $b = 6$, and the equation $a + b + c = 0$ gives $a = -9/2$.

2. To extend this linearly independent set to a basis, it suffices to add any vector outside of $\text{span}(S_2)$. Notice that the first two coordinates of each element in $S_2$ are equal. Thus any linear combination of elements in $S_2$ will have the first two coordinates equal. So, the vector $(1, 0, 0)$, for example, does not lie in $\text{span}(S_2)$, and so extends $S_2$ to a basis. By inspection, $(3, 3, 5) = (1, 1, 1) + 2(1, 1, 2)$. 


Question 3

\( N(T) = \{ f \in \mathbb{P}_2(\mathbb{R}) : (x-1)f = 0 \} = \{0\} \), as the degree of \((x-1)f(x)\) is at least one unless \(f(x) = 0\). Therefore \(T\) is one-to-one and has \(\text{nullity}(T) = 0\) and \(\text{rank}(T) = 3\). This implies that the images of vectors forming a basis for \(\mathbb{P}_2(\mathbb{R})\) will be a basis for the range (they will generate, and there are the correct number of them). Therefore \(\{T(1), T(x), T(x^2)\} = \{(x-1), (x-1)x, (x-1)x^2\}\) is a basis for the range of \(T\).

Question 4

(1) If \(e^x\) and \(xe^x\) were linearly dependent, there would exist some scalar \(c \in \mathbb{R}\setminus\{0\}\) such that \(cxe^x = e^x\). As the right-hand side is always positive, and the left-hand side can be negative regardless of the value of \(c\), this is impossible.

(2) Let \(f(x) = ae^x + bxe^x\) be an element of \(V\). Then \(f(x) \in N\left(\frac{d}{dx}\right)\) if and only if

\[
\frac{df}{dx} = ae^x + bxe^x = 0 = (a + b)e^x + bxe^x = 0.
\]

Since \(e^x\) and \(xe^x\) are linearly independent, we see that \(f(x) \in N\left(\frac{d}{dx}\right)\) if and only if \(a + b = 0\) and \(b = 0\). As the only solutions are \(a = b = 0\), we have that \(N\left(\frac{d}{dx}\right) = 0\).

Once again, \(\frac{d}{dx} : V \to V\) is one-to-one, has nullity 0, rank 2, and a basis for the range is given by the images of the basis vectors of the domain, i.e. \(\{e^x, e^x + xe^x\}\). But since \(\text{span}\{e^x, e^x + xe^x\} = \text{span}\{e^x, xe^x\} = V\), we see that this map is also onto.

(3) As we have \(\frac{d}{dx}e^x = e^x\) and \(\frac{d}{dx}xe^x = e^x + xe^x\), the matrix is \(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\).

Question 5

Let \(\varepsilon = \{e_1, e_2, e_3, e_4, e_5\}\) be the standard ordered basis of \(\mathbb{R}^5\) and let \(\gamma = \{\gamma_1, \ldots, \gamma_5\} = \{e_2, e_4, e_5, e_1, e_3\}\) be a different ordered basis, just given by a permutation of the basis vectors. Then \(Q\) is the change of basis matrix \(I_{\varepsilon}^\gamma\), and so \(Q^{-1}AQ\) will simply be \([L_A]_\gamma\) (see Theorem 2.23 and its Corollary).

So we need to compute the matrix representation \([L_A]_\gamma\). For simplicity of notation, let \(v = e_1 + e_2 + e_3 + e_4 + e_5\) (represented by the vector of all 1’s). Then \(A(\gamma_1) = A(e_2) = v - e_3 = v - \gamma_5\), so the first column of \([L_A]_\gamma\) is

\[
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.
\]

Similarly we compute

\[
[L_A]_\gamma = \begin{pmatrix}
1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1
\end{pmatrix}.
\]