Problem Set # 6 (due in class Thursday March 1st)

Reading: FIS 2.4, 2.5, 3.1

Problems:

1. FIS 2.4 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 9 (By “arbitrary matrices” they mean matrices that are not necessarily square), 10 (The definition of a matrix $A$ being invertible is that there exists $B$ such that both $AB$ and $BA$ are the identity, in this problem you prove that you only need to know one of these), 15, 17 (For the first part, use the restriction of $T$ to $V_0$; for the second part, use exercise 15), 20.

2. FIS 2.5 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 2d, 3b, 6bd, 13.

3. FIS 3.1 Exercises 1 (If true, cite or prove it; if false, give a counterexample), 3c, 9.

4. Let $F$ be a field and consider the matrix

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $M_{2 \times 2}(F)$.

(a) Prove that $M$ is invertible if and only if $ad - bc \neq 0$, in which case

$$M^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

(Hint. Use the work you did on Problem Set #2.)

(b) Write down all invertible matrices in $M_{2 \times 2}(\mathbb{F}_2)$.

5. Let $V$ be an $\mathbb{F}_p$-vector space of dimension $n$.

(a) Calculate the number of vectors in $V$. This is a function of $p$ and $n$.

(Hint: You can assume that $V = \mathbb{F}_p^n$. Why?)

(b) For each $1 \leq k \leq n$, calculate the number of $k$-tuples of linearly independent vectors in $V$. This is a function of $p$, $n$, and $k$.

(Hint: Start with a non-zero vector $v_1$, then choose $v_2$ not in the span of $\{v_1\}$, then choose $v_3$ not in the span of $\{v_1, v_2\}$, and so on, using the fact that you know the size of the span by the previous part.)

(c) Calculate the number of invertible $n \times n$ matrices over $\mathbb{F}_p$.

(Hint: Use the previous part.)