Notation: Let $V$ be an $F$-vector space. A linear transformation $T : V \to V$ is often called a linear operator on $V$. For $n > 0$, we write $T^n$ for $T$ composed with itself $n$ times. For a matrix $A \in M_{m \times n}(F)$, the left multiplication transformation is the linear map $L_A : F^n \to F^m$ defined by $L_A(v) = Av$, where we consider $v$ as a column vector (or rather, as an $n \times 1$ matrix) and $Av$ is the product of $A$ and $v$.

Reading: FIS 2.2, 2.3

Problems:

1. FIS 2.2 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 2bce, 4, 5acdfg, 8, 9, 11 (Hint: Use §1.6 Corollary 2 part c), 13, 14.

2. FIS 2.3 Exercises 1 (If true, then either cite or prove it, if false then provide a counterexample), 4acd, 9, 11, 12 (For the second question in each part, if true, prove it, and if false then provide a counterexample), 16.

3. Let $V$ be a vector space and $T : V \to V$ a linear operator.
   (1) Prove that $T = T^2$ if and only if there exist subspaces $W_0, W_1$ of $V$ and an internal direct sum decomposition $V = W_0 \oplus W_1$ such that $T$ restricted to $W_0$ is the zero map and $T$ restricted to $W_1$ is the identity map.
   (2) Assume that $V$ is finite dimensional. Prove that $T = T^2$ if and only if there exists an ordered basis $\beta$ such that $[T]_\beta$ is a diagonal matrix whose diagonal entries are either 0 or 1. Hint. FIS 2.3 exercises 16 and (the hint in) 17 will come in handy.

4. For $\theta \in \mathbb{R}$, consider the matrix $T_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \in M_{2 \times 2}(\mathbb{R})$
   (1) For any $\theta$, verify that $T_\theta$ is invertible and that $T_\theta^{-1} = T_{-\theta}$.
   (2) Prove that $L_{T_\theta} : \mathbb{R}^2 \to \mathbb{R}^2$ is counter-clockwise rotation by angle $\theta$.
   (Hint: Calculate how the slope of a nonzero vector changes.)