

FIBERED FACES, VEERING TRIANGULATIONS, AND THE ARC COMPLEX

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ABSTRACT. We study the connections between subsurface projections in curve and arc complexes in fibered 3-manifolds and Agol’s veering triangulation. The main theme is that large-distance subsurfaces in fibers are associated to large simplicial regions in the veering triangulation, and this correspondence holds uniformly for all fibers in a given fibered face of the Thurston norm.

1. INTRODUCTION

Let M be a 3-manifold fibering over the circle with fiber S and pseudo-Anosov monodromy f . The stable/unstable laminations λ^+, λ^- of f give rise to a function on the essential subsurfaces of S ,

$$Y \mapsto d_Y(\lambda^+, \lambda^-),$$

where d_Y denotes distance in the curve and arc complex of Y between the lifts of λ^\pm to the cover of S homeomorphic to Y . This distance function plays an important role in the geometry of the mapping class group of S [MM00, BKMM12, MS13], and in the hyperbolic geometry of the manifold M [Min10, BCM12].

In this paper we study the function d_Y when M is fixed and the fibration is varied. The fibrations of a given manifold are organized by the faces of the unit ball of Thurston’s norm on $H_2(M, \partial M)$, where each *fibered face* \mathcal{F} has the property that every integral class in the cone $\mathbb{R}_+\mathcal{F}$ represents a fiber. There is a pseudo-Anosov flow which is transverse to every fiber represented by \mathcal{F} , and whose stable/unstable laminations $\Lambda^\pm \subset M$ intersect each fiber to give the laminations associated to its monodromy. With this we note that the distance $d_Y(\lambda^+, \lambda^-)$ can be defined for any subsurface Y of any fiber in \mathcal{F} . We use $d_Y(\Lambda^+, \Lambda^-)$ to denote this quantity.

Our main results give explicit connections between d_Y and the *veering triangulation* of M , introduced by Agol [Ago11] and refined by Guéritaud [Gué15], with the main feature being that when d_Y satisfies explicit lower bounds, Y corresponds to an embedded subcomplex of the veering triangulation. In this way, the “complexity” of the monodromy f is visible directly in the triangulation in a way that is independent of the choice of fiber in the face \mathcal{F} . This is in contrast with the results of [BCM12] in which the estimates relating d_Y to the hyperbolic geometry of M are heavily dependent on the genus of the fiber.

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The results are cleanest in the setting of a *fully-punctured* fiber, that is when the singularities of the monodromy f are assumed to be punctures of the surface S (one can obtain such examples by starting with any M and removing the punctures and their flow orbits). All fibers in a face \mathcal{F} are fully-punctured when any one is, and in this case we say that \mathcal{F} is a *fully-punctured face*. In this setting M is a cusped manifold and the veering triangulation τ is an ideal triangulation.

We obtain bounds on $d_Y(\Lambda^+, \Lambda^-)$ that hold for Y in any fiber of a given fibered face:

Theorem 1.1 (Bounding projections over a fibered face). *Let M be a hyperbolic 3-manifold with fully-punctured fibered face \mathcal{F} and veering triangulation τ . For any subsurface W of any fiber of \mathcal{F} ,*

$$\alpha \cdot (d_W(\Lambda^-, \Lambda^+) - \beta) < |\tau|,$$

where $|\tau|$ is the number of tetrahedra in τ , $\alpha = 1$ and $\beta = 10$ when W is an annulus and $\alpha = 3|\chi(W)|$ and $\beta = 8$ when W is not an annulus.

Note that this means that there is an explicit constant $c(M)$ so that there are at most finitely many subsurfaces W with $d_W \geq c$, no matter which fiber they might be in. Further, the complexity $|\chi(W)|$ of any subsurface W of any fiber of \mathcal{F} with $d_W(\Lambda^+, \Lambda^-) \geq 9$ is also uniformly bounded.

In addition, given one fiber with a collection of subsurfaces of large d_Y , we obtain control over the appearance of high-distance subsurfaces in all other fibers:

Theorem 1.2 (Subsurface dichotomy). *Let M be a hyperbolic 3-manifold with fully-punctured fibered face \mathcal{F} and suppose that S and F are each fibers in $\mathbb{R}_+\mathcal{F}$. If W is a subsurface of F , then either W is isotopic along the flow to a subsurface of S , or*

$$3|\chi(S)| \geq d_W(\Lambda^-, \Lambda^+) - \beta,$$

where $\beta = 10$ if W is an annulus and $\beta = 8$ otherwise.

One can apply this theorem with S taken to be the smallest-complexity fiber in \mathcal{F} . In this case there is some finite list of “large” subsurfaces of S , and for all other fibers and all subsurfaces W with d_W sufficiently large, W is already accounted for on this finite list.

For a sample application of [Theorem 1.2](#), let W be an essential annulus with core curve w in a fiber F of M and suppose that $d_W(\Lambda^-, \Lambda^+) \geq K - 10 > 0$. (We note that it is easy to construct explicit examples of M with $d_W(\Lambda^-, \Lambda^+)$ as large as one wishes by starting with a pseudo-Anosov homeomorphism of F with large twisting about the curve w .) If w is trivial in $H_1(M)$, then [Theorem 1.2](#) (or more precisely [Corollary 6.7](#)) implies that w is actually isotopic to a simple closed curve in every fiber in the cone $\mathbb{R}_+\mathcal{F}$ containing F . When w is nontrivial in $H_1(M)$ it determines a codimension-1 hyperplane P_w in $H^1(M) = H_2(M, \partial M)$ consisting of cohomology classes which vanish on w . For each fiber S of $\mathbb{R}_+\mathcal{F}$ either S is contained in P_w , in which case w is isotopic to a simple closed curve in S as before, or S lies outside of P_w and $|\chi(S)| \geq K/3$. We remark that the second alternative is non-vacuous so long as $H^1(M)$ has rank at least 2.

The general (non-fully-punctured) setting is also approachable with our techniques, but a number of complications arise and the connection to the veering triangulation is much less explicit. An extension of the results in this paper to the general setting will be the subject of a subsequent paper.

Pockets in the veering triangulation. When Y is a subsurface of a fiber X in \mathcal{F} and $d_Y(\Lambda^+, \Lambda^-) > 1$, we show ([Theorem 5.2](#)) that Y is realized simplicially in the veering triangulation lifted to the cover $X \times \mathbb{R}$. If $d_Y(\Lambda^+, \Lambda^-)$ is even larger then this realization can be thickened to a “pocket”, which is a simplicial region bounded by two isotopic copies of Y . With sufficiently large assumptions this pocket can be made to embed in M as well, and this is our main tool for connecting arc complexes to the veering triangulation and establishing [Theorems 1.1](#) and [1.2](#):

Theorem 1.3. *Suppose Y is a subsurface of a fiber X with $d_Y(\lambda^-, \lambda^+) > \beta$, where $\beta = 8$ if Y is nonannular and $\beta = 10$ if Y is an annulus. Then there is an embedded simplicial pocket V in M isotopic to a thickening of Y , and with $d_Y(V^-, V^+) \geq d_Y(\lambda^-, \lambda^+) - \beta$.*

In this statement, V^+ and V^- refer to the triangulations of the top and bottom surfaces of the pockets, regarded as simplicies in the curve and arc complex $\mathcal{A}(Y)$.

The veering triangulation in fact recovers a number of aspects of the geometry of curve and arc complexes in a fairly concrete way. As an illustration we prove

Theorem 1.4. *In the fully punctured setting, the arcs of the veering triangulation form a totally geodesic subset of the curve and arc complex.*

Hierarchies of pockets. One is naturally led to generalize [Theorem 1.3](#) from a result embedding one pocket at a time to a description of all pockets at once. Indeed [Proposition 6.5](#) tells us that whenever subsurfaces Y and Z of X have large enough projection distances and are either disjoint or overlapping, they have associated pockets V_Y and V_Z which are disjoint in $X \times \mathbb{R}$. These facts, taken together with [Theorem 1.4](#), strongly suggest that the veering triangulation τ encodes the hierarchy of curve complex geodesics between λ^\pm as introduced by Masur-Minsky in [[MM00](#)]. We expect that, using a version of [Theorem 1.4](#) that applies to subsurfaces and adapting the notion of “tight geodesic” from [[MM00](#)], one can carry out a hierarchy-like construction within the veering triangulation and recover much of the structure found in [[MM00](#)], with more concrete control, at least in the fully-punctured setting. We plan to explore this approach in future work.

Related and motivating work. The theme of using fibered 3-manifolds to study infinite families of monodromy maps is deeply explored in McMullen [[McM00](#)] and Farb-Leininger-Margalit [[FLM11](#)], where the focus is on Teichmüller translation distance.

Distance inequalities analogous to [Theorem 1.2](#), in the setting of Heegaard splittings, appear in Hartshorn [[Har02](#)], and then more fully in Scharlemann-Tomova [[ST06](#)]. Bachman-Schleimer [[BS05](#)] use Heegaard surfaces to give bounds on the

curve-complex translation distance of the monodromy of a fibering. All of these bounds apply to entire surfaces and not to subsurface projections. In Johnson-Minsky-Moriah [JMM10], subsurface projections are considered in the setting of Heegaard splittings. A basic difficulty in these papers which we do not encounter is the compressibility of the Heegaard surfaces, which makes it tricky to control essential intersections. On the other hand, unlike the surfaces and handlebodies that are used to obtain control in the Heegaard setting, the foliations we consider here are infinite objects, and the connection between them and finite arc systems in the surface is a priori dependent on the fiber complexity. The veering triangulation provides a finite object that captures this connection in a more uniform way.

The totally-geodesic statement of [Theorem 1.4](#) should be compared to [Theorem 1.2](#) of Tang-Webb [TW15], in which Teichmüller disks give rise to quasi-convex sets in curve complexes. While the results of Tang-Webb are more general, they are coarse, and it is interesting that in our setting a tighter statement holds.

Summary of the paper. In [Section 2](#) we set some notation and give Guéritaud’s construction of the veering triangulation. We also recall basic facts about curve and arc complexes, subsurface projections and Thurston’s norm on homology. We spend some time in this section describing the flat geometry of a punctured surface with an integrable holomorphic quadratic differential, and in particular giving an explicit description of the circle at infinity of its universal cover ([Proposition 2.2](#)). While this is a fairly familiar picture, some delicate issues arise because of the incompleteness of the metric at the punctures.

In [Section 3](#) we study *sections* of the veering triangulations, which are simplicial surfaces isotopic to X in the cover $X \times \mathbb{R}$, and transverse to the suspension flow of the monodromy. These can be thought of as triangulations of the surface X using only edges coming from the veering triangulation. We prove [Lemma 3.2](#) which says that a partial triangulation of X can always be extended to a full section, and [Proposition 3.3](#) which says that any two extensions of a partial triangulation are connected by a sequence of “tetrahedron moves”. This is what allows us to define and study the “pockets” that arise between any two sections.

In [Section 4](#) we define a simple but useful construction, rectangle and triangle hulls, which map saddle connections in our surface to unions of edges of the veering triangulations. An immediate consequence of the properties of these hulls is a proof of [Theorem 1.4](#).

In [Section 5](#) we apply the flat geometry developed in [Section 2](#) to control the convex hulls of subsurfaces of the fiber, and then use [Section 4](#) to construct what we call τ -hulls, which are representatives of the homotopy class of a subsurface that are simplicial with respect to the veering triangulations. [Theorem 5.2](#) states that quite mild assumptions on $d_Y(\lambda^+, \lambda^-)$ imply that its τ -hull has embedded interior. The idea here is that any pinching point of the hull is crossed by leaves of λ^+ and λ^- that intersect each other very little. The main results of both [Section 4](#) and [Section 5](#) apply in a general setting and do not require that the surface X is fully-punctured.

In [Section 6](#) we put these ideas together to prove our main theorems for fibered manifolds with a fully-punctured fibered face. In [Proposition 6.2](#) we describe the

maximal pocket associated to a subsurface Y with $d_Y(\Lambda^+, \Lambda^-)$ sufficiently large (greater than 2, for nonannular Y). We then introduce the notion of an *isolated pocket*, which is a subpocket of the maximal pocket that has good embedding properties in the manifold M . The existence and embedding properties of these pockets are established in [Lemma 6.4](#) and [Proposition 6.5](#), which together allow us to prove [Theorem 1.3](#).

From here, a simple counting argument gives [Theorem 1.1](#): the size of the embedded isolated pockets is bounded from below in terms of $d_Y(\Lambda^+, \Lambda^-)$ and $\chi(Y)$, and from above by the total number of veering tetrahedra.

To obtain [Theorem 1.2](#), we use the pocket embedding results to show that, if Y is a subsurface of one fiber F that essentially intersects another fiber S , then S must cross every level surface of the isolated pocket of Y , and hence the complexity of S gives an upper bound for $d_Y(\Lambda^+, \Lambda^-)$. To complete the proof we need to show that, if Y does not essentially cross S , it must be isotopic to an embedded (and not merely immersed) subsurface of S . This is handled by [Lemma 6.6](#), which may be of independent interest. It gives a uniform upper bound for $d_Y(\Lambda^+, \Lambda^-)$ when Y corresponds to a finitely generated subgroup of $\pi_1(S)$, unless Y covers an embedded subsurface.

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2. BACKGROUND

The following notation will hold throughout the paper. Let \bar{X} be a closed Riemann surface with an integrable meromorphic quadratic differential q (which may have poles of order 1). We denote the vertical and horizontal foliations of q by λ^+ and λ^- respectively. Let \mathcal{P} be a finite subset of \bar{X} that includes the poles of q if any, and let $X = \bar{X} \setminus \mathcal{P}$. Let $\text{sing}(q)$ denote the singularities of q , which we take by definition to include the punctures \mathcal{P} . We require further that q has no horizontal or vertical saddle connections, that is no leaves of λ^\pm that connect two points of $\text{sing}(q)$. This situation holds in particular if λ^\pm are the stable/unstable foliations of a pseudo-Anosov map $f : X \rightarrow X$, which will often be the case for us. If $\mathcal{P} = \text{sing}(q)$ we say X is *fully-punctured*.

Let \hat{X} denote the metric completion of the universal cover \tilde{X} of X , and note that there is an infinite branched cover $\hat{X} \rightarrow \bar{X}$, branched over the points of \mathcal{P} . The preimage $\hat{\mathcal{P}}$ of \mathcal{P} is the set of completion points. The space \hat{X} is a complete CAT(0) space with the metric induced by q .

2.1. Veering triangulations. In this section let $\mathcal{P} = \text{sing}(q)$. The veering triangulation, originally defined by Agol in [\[Ago11\]](#) in the case where q corresponds to a pseudo-Anosov $f : X \rightarrow X$, is an ideal layered triangulation of $X \times \mathbb{R}$ which projects to a triangulation of the mapping torus M of f . The definition we give here is due to Guéritaud [\[Gué15\]](#). (Agol’s “veering” property itself will not actually play a role in this paper, so we will not give its definition).

A *singularity-free rectangle* in \hat{X} is an embedded rectangle whose edges are leaf segments of the lifts of λ^\pm and whose interior contains no singularities of \hat{X} . If R is a *maximal* singularity-free rectangle in \hat{X} then it must contain a singularity on each edge. Note that there cannot be more than one singularity on an edge since λ^\pm have no saddle connections. We associate to R an ideal tetrahedron whose vertices are $\partial R \cap \hat{\mathcal{P}}$, as in [Figure 1](#). This tetrahedron comes equipped with a map into \hat{X} as pictured.

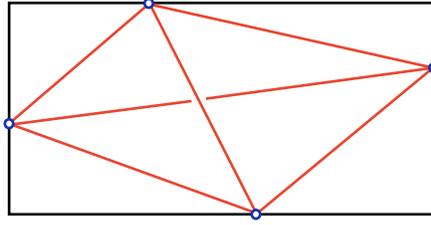


FIGURE 1. A maximal singularity-free rectangle R defines a tetrahedron equipped with a map into R .

The tetrahedron comes with a natural orientation, inherited from the orientation of \hat{X} using the convention that the edge connecting the horizontal boundaries of the rectangle lies *above* the edge connecting the vertical boundaries.

The union of all these ideal tetrahedra, with faces identified whenever they map to the same triangle in \hat{X} , is Guéritaud's construction of the veering triangulation of $\tilde{X} \times \mathbb{R}$.

Theorem 2.1. [[Gué15](#)] *The complex of tetrahedra associated to maximal rectangles of q is an ideal triangulation $\tilde{\tau}$ of $\tilde{X} \times \mathbb{R}$, and the maps of tetrahedra to their defining rectangles piece together to a fibration $\pi : \tilde{X} \times \mathbb{R} \rightarrow \tilde{X}$. The action of $\pi_1(X)$ on (\tilde{X}, \tilde{q}) lifts simplicially to $\tilde{\tau}$, and equivariantly with respect to π . The quotient is a triangulation of $X \times \mathbb{R}$.*

If q corresponds to a pseudo-Anosov $f : X \rightarrow X$ then the action of f on (X, q) lifts simplicially and π -equivariantly to $\Phi : X \times \mathbb{R} \rightarrow X \times \mathbb{R}$. The quotient is a triangulation τ of the mapping torus M . The fibers of π descend to flow lines for the suspension flow of f .

We will frequently abuse notation and use τ to refer to the triangulation both in M and in its covers.

We note that a saddle connection σ of q is an edge of τ if and only if σ spans a singularity-free rectangle in X . See [Figure 2](#).

If e and f are two crossing τ edges with rectangles R_e and R_f , note that R_e crosses R_f from top to bottom, or from left to right – any other configuration would contradict the singularity-free property of the rectangles ([Figure 3](#)). If $\text{slope}(e)$ denotes the absolute value of the slope of e with respect to q , we can see that R_e crosses R_f from top to bottom if and only if $\text{slope}(e) > \text{slope}(f)$. We say that e is

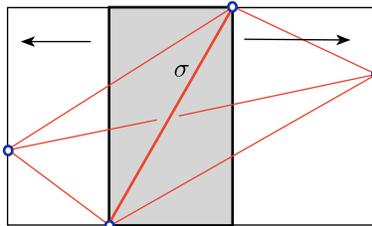


FIGURE 2. The singularity-free rectangle spanned by σ can be extended horizontally (or vertically) to a maximal one.

more vertical than f and also write $e > f$. We will see that $e > f$ corresponds to e lying higher than f in the flow direction.

Indeed we can see already that the relation $>$ is transitive, since if $e > f$ and $f > g$ then the rectangle of g is forced to intersect the rectangle of e .

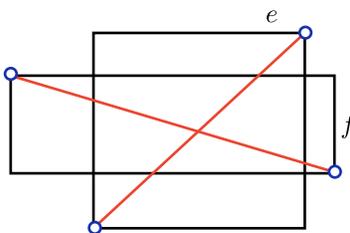


FIGURE 3. The rectangle of e crosses f from top to bottom and we write $e > f$.

We conclude with a brief description of the local structure of τ around an edge e : The rectangle spanned by e can be extended horizontally to define a tetrahedron lying below e in the flow direction, and vertically to define a tetrahedron lying above e in the flow direction. Call these Q_- and Q_+ as in Figure 4. Between these, on each side of e , is a sequence of tetrahedra Q_1, \dots, Q_m so that two successive tetrahedra in the sequence $Q_-, Q_1, \dots, Q_m, Q_+$ share a triangular face adjacent to e . We find this sequence by starting with one triangular face of the first quadrilateral and extending its spanning rectangle as far as possible in each possible way. Figure 4 illustrates this structure on one side of an edge e . In particular note that the link of an edge is a circle, as expected.

2.2. Arc and curve complexes. The arc and curve complex $\mathcal{A}(Y)$ for a compact surface Y is usually defined as follows: its vertices are essential homotopy classes of embedded circles and properly embedded arcs $([0, 1], \{0, 1\}) \rightarrow (Y, \partial Y)$, where “essential” means not homotopic to a point or into the boundary [MM00]. We must be clear about the meaning of homotopy classes here, for the case of arcs: If Y is not an annulus, homotopies of arcs are assumed to be homotopies of maps of pairs. When Y is an annulus the homotopies are also required to fix the endpoints. Simplices

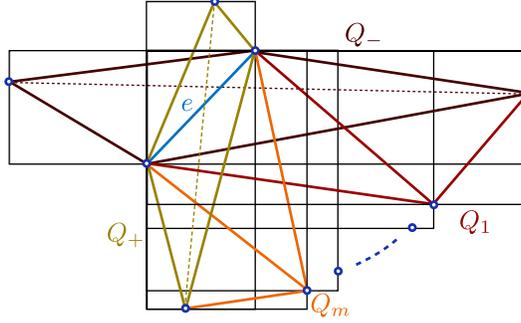


FIGURE 4. The tetrahedra adjacent to an edge e on one side form a sequence “swinging” around e

of $\mathcal{A}(Y)$, in all cases, correspond to tuples of vertices which can be simultaneously realized by maps that are disjoint on their interiors.

It will be useful, in the non-annular case, to observe that the following definition is equivalent: Instead of maps of closed intervals consider proper embeddings $\mathbb{R} \rightarrow \text{int}(Y)$ into the interior of Y , with equivalence arising from proper homotopy. This definition is independent of the compactification of $\text{int}(Y)$. The natural isomorphism between these two versions of $\mathcal{A}(Y)$ is induced by a straightening construction in a collar neighborhood of the boundary.

If $Y \subset S$ is an essential subsurface (meaning the inclusion of Y is π_1 -injective and is not homotopic to a point or to an end of S), we have subsurface projections $\pi_Y(\lambda)$ which are defined for simplices $\lambda \subset \mathcal{A}(S)$ that intersect Y essentially, as well as for geodesic laminations in S which intersect Y essentially. Namely, after lifting λ to the cover S_Y associated to $\pi_1(Y)$, we obtain a collection of properly embedded disjoint essential arcs and curves, which determine a simplex of $\mathcal{A}(Y)$. We let $\pi_Y(\lambda)$ be the union of these vertices [MM00]. Note that when Y is an annulus these arcs have natural endpoints coming from the standard compactification of $\tilde{S} = \mathbb{H}^2$ by a circle at infinity.

When Y is not an annulus and λ and ∂Y are in minimal position, we can also identify $\pi_Y(\lambda)$ with the isotopy classes of components of $\lambda \cap \partial Y$.

These definitions naturally extend to immersed surfaces arising from covers of S . Let Γ be a finitely generated subgroup of $\pi_1(S)$. Then the corresponding cover $S_\Gamma \rightarrow S$ has a compact core W – a compact subsurface $W \subset S_\Gamma$ such that $S_\Gamma \setminus W$ is a collection of boundary parallel annuli. For curves or laminations λ^\pm of S , we have lifts $\tilde{\lambda}^\pm$ to S_Γ and define $d_W(\lambda^-, \lambda^+) = d_{S_\Gamma}(\tilde{\lambda}^-, \tilde{\lambda}^+)$.

Throughout this paper, when λ, λ' are two laminations or arc/curve systems, we denote by $d_Y(\lambda, \lambda')$ the *minimal* distance between their images in $\mathcal{A}(Y)$, that is

$$d_Y(\lambda, \lambda') = \min\{d_Y(l, l') : l \in \pi_Y(\lambda), l' \in \pi_Y(\lambda')\}.$$

To denote the *maximal* distance between λ and λ' in $\mathcal{A}(Y)$ we write

$$\text{diam}_Y(\lambda, \lambda') = \text{diam}_{\mathcal{A}(Y)}(\pi_Y(\lambda) \cup \pi_Y(\lambda')).$$

2.3. Flat geometry. In this section we return to the geometry of (X, q) and describe a circle at infinity for the flat metric induced by q on the universal cover \tilde{X} . Because of incompleteness at the punctures \mathcal{P} (which in this section can be arbitrary), the connection between this and the usual circle at infinity for \mathbb{H}^2 requires a bit of care. A related discussion appears in Guéritaud [Gué15], although he deals explicitly only with the fully-punctured case. With this picture of the circle at infinity we will be able to describe π_Y in terms of q -geodesic representatives, and to describe a q -convex hull for essential subsurfaces of X .

The completion points $\hat{\mathcal{P}}$ in \hat{X} correspond to parabolic fixed points for $\pi_1(X)$ in $\partial\mathbb{H}^2$, and we abuse notation slightly by identifying $\hat{\mathcal{P}}$ with this subset of $\partial\mathbb{H}^2$.

A *complete q -geodesic ray* is either a geodesic ray $r : [0, \infty) \rightarrow \hat{X}$ of infinite length, or a finite-length geodesic segment that terminates in $\hat{\mathcal{P}}$. A complete q -geodesic line is a geodesic which is split by any point into two complete q -geodesic rays. Our goal in this section is to describe a circle at infinity that corresponds to endpoints of these rays.

Proposition 2.2. *There is a compactification $\beta(\tilde{X})$ on which $\pi_1(X)$ acts by homeomorphisms, with the following properties:*

- (1) *There is a $\pi_1(X)$ -equivariant homeomorphism $\beta(\tilde{X}) \rightarrow \overline{\mathbb{H}^2}$, extending the identification of \tilde{X} with \mathbb{H}^2 and taking $\hat{\mathcal{P}}$ to the corresponding parabolic fixed points in $\partial\mathbb{H}^2$.*
- (2) *If l is a complete q -geodesic line in \hat{X} then its image in $\overline{\mathbb{H}^2}$ is an embedded arc with endpoints on $\partial\mathbb{H}^2$. Conversely, every pair of distinct points x, y in $\partial\beta(\tilde{X}) = \beta(\tilde{X}) \setminus \tilde{X}$ are the endpoints of a complete q -geodesic line. The termination point in $\partial\mathbb{H}^2$ of a complete q -geodesic ray is in $\hat{\mathcal{P}}$ if and only if it has finite length.*

One of the tricky points of this picture is that q -geodesic rays and lines may meet points of the boundary $\partial\beta(\tilde{X})$ not just at their endpoints.

Proof. When $\mathcal{P} = \emptyset$ and X is a closed surface, \tilde{X} is quasi-isometric to \mathbb{H}^2 and the proposition holds for the standard Gromov compactification. We assume from now on that $\mathcal{P} \neq \emptyset$.

We begin by setting $\hat{\mathbb{H}}^2 = \mathbb{H}^2 \cup \hat{\mathcal{P}}$ and endowing it with the topology obtained by taking, for each $p \in \hat{\mathcal{P}}$, horoballs based at p as a neighborhood basis for p .

Lemma 2.3. *The natural identification of \tilde{X} with \mathbb{H}^2 extends to a homeomorphism from \hat{X} to $\hat{\mathbb{H}}^2$.*

Proof. First note that $\hat{\mathcal{P}}$ is discrete as both a subspace of \hat{X} and of $\hat{\mathbb{H}}^2$. Hence, it suffices to show that a sequence of points x_i in $\tilde{X} = \mathbb{H}^2$ converges to a point $p \in \hat{\mathcal{P}}$ in \hat{X} if and only if it converges to p in $\hat{\mathbb{H}}^2$. This follows from the fact that the horoball neighborhoods of p descend to cusp neighborhoods in X which form a neighborhood basis for the puncture that is equivalent to the neighborhood basis of q -metric balls. \square

Our strategy now is to form the *Freudenthal space* of \hat{X} and equivalently $\hat{\mathbb{H}}^2$, which appends a space of *ends*. This space will be compact but not Hausdorff, and after a mild quotient we will obtain the desired compactification which can be identified with $\overline{\mathbb{H}^2}$. Simple properties of this construction will then allow us to obtain the geometric conclusions in part (2) of the proposition.

Let $\epsilon(\hat{X})$ be the space of ends of \hat{X} , that is the inverse limit of the system of path components of complements of compact sets in \hat{X} . The Freudenthal space $\text{Fr}(\hat{X})$ is the union $\hat{X} \cup \epsilon(\hat{X})$ endowed with the topology generated by using path components of complements of compacta to describe neighborhood bases for the ends. Because \hat{X} is not locally compact, $\text{Fr}(\hat{X})$ is not guaranteed to be compact, and we have to take a bit of care to describe it.

The construction can of course be repeated for $\hat{\mathbb{H}}^2$, and the homeomorphism of [Lemma 2.3](#) gives rise to a homeomorphism $\text{Fr}(\hat{X}) \rightarrow \text{Fr}(\hat{\mathbb{H}}^2)$. Let us work in $\hat{\mathbb{H}}^2$ now, where we can describe the ends concretely using the following observations:

Every compact set $K \subset \hat{\mathbb{H}}^2$ meets $\hat{\mathcal{P}}$ in a finite set A (since $\hat{\mathcal{P}}$ is discrete in $\hat{\mathbb{H}}^2$), and such a K is contained in an embedded closed disk D which also meets $\hat{\mathcal{P}}$ at A . (This is not hard to see but does require attention to deal correctly with the horoball neighborhood bases). The components of $\hat{\mathbb{H}}^2 \setminus D$ determine a partition of $\epsilon(\hat{\mathbb{H}}^2)$, which in fact depends only on the set A and not on D (if D' is another disk meeting $\hat{\mathcal{P}}$ at A , then $D \cup D'$ is contained in a third disk D'' , and this common refinement of the neighborhoods gives the same partition). Thus we have a more manageable (countable) inverse system of neighborhoods in $\epsilon(\hat{\mathbb{H}}^2)$, and with this description it is not hard to see that $\epsilon(\hat{\mathbb{H}}^2)$ is a Cantor set.

For each $p \in \hat{\mathcal{P}}$ there are two distinguished ends $p^+, p^- \in \epsilon(\hat{\mathbb{H}}^2)$ defined as follows: For each finite subset $A \subset \hat{\mathcal{P}}$ with at least two points one of which is p , the two elements adjacent to p in the circle (or equivalently, in the boundary of any $D \subset \hat{\mathbb{H}}^2$ meeting $\hat{\mathcal{P}}$ in A) define neighborhoods in $\epsilon(\hat{\mathbb{H}}^2)$, and this pair of neighborhood systems determines p^+ and p^- respectively.

One can also see that p^+ (and p^-) and p do not admit disjoint neighborhoods, and this is why $\text{Fr}(\hat{\mathbb{H}}^2)$ is not Hausdorff. We are therefore led to define the quotient space

$$\beta(\hat{\mathbb{H}}^2) = \text{Fr}(\hat{\mathbb{H}}^2) / \sim,$$

where we make the identifications $p^- \sim p \sim p^+$, for each $p \in \hat{\mathcal{P}}$.

We can make the same definitions in \hat{X} , obtaining

$$\beta(\hat{X}) = \text{Fr}(\hat{X}) / \sim,$$

which we rename $\beta(\tilde{X})$ to remind ourselves that it depends only on the original \tilde{X} and its metric q . Since the definitions are purely in terms of the topology of the spaces $\hat{\mathbb{H}}^2$ and \hat{X} , the homeomorphism of [Lemma 2.3](#) extends to a homeomorphism $\beta(\tilde{X}) \rightarrow \beta(\hat{\mathbb{H}}^2)$.

Part (1) of [Proposition 2.2](#) follows once we establish that the identity map of \mathbb{H}^2 extends to a homeomorphism

$$\beta(\hat{\mathbb{H}}^2) \cong \overline{\mathbb{H}^2}.$$

This is not hard to see once we observe that the disks used above to define neighborhood systems can be chosen to be ideal hyperbolic polygons. Their halfspace complements serve as neighborhood systems for points of $\partial\mathbb{H}^2 \setminus \hat{\mathcal{P}}$. A sequence converges in $\overline{\mathbb{H}^2}$ to a point $p \in \hat{\mathcal{P}}$ either if it is eventually contained in any horoball, or in infinitely many halfspaces adjacent to p on one side or the other. This is modeled exactly by the equivalence relation \sim .

For part (2), let D_0 be a fundamental domain for $\pi_1(X)$ in \hat{X} , which may be chosen to be a disk with vertices at points of $\hat{\mathcal{P}}$, and of finite q diameter. Translates of D_0 can be used to build a sequence of nested disks D_n exhausting \hat{X} , each of which meets $\hat{\mathcal{P}}$ in a finite set of vertices, and whose boundary is composed of arcs of bounded diameter between successive vertices.

A complete geodesic ray r either has finite length and terminates in a point of $\hat{\mathcal{P}}$, or has infinite length in which case it leaves every compact set of \hat{X} , and visits each point of $\hat{\mathcal{P}}$ at most once. Thus it must terminate in a point of $\epsilon(\hat{X})$ in the Freudenthal space. We claim that this point cannot be p^+ or p^- for $p \in \hat{\mathcal{P}}$. If r terminates in p_+ , then for each disk D_n it must pass through the edge of ∂D_n adjacent to p on the side associated to p^+ . Any two such edges meet p at one of finitely many angles (images of corners of D_0), and hence the accumulated angle between edges goes to ∞ with n . If we replace these edges by their q -geodesic representatives, the angles still go to ∞ . This means that we obtain, infinitely often, segments of r whose endpoints are a bounded distance from p and which are connected to p by geodesic segments meeting at angle greater than π . But q is a CAT(0) metric so this can only happen if each segment of r passes through p . This contradicts the fact that r can visit each point of $\hat{\mathcal{P}}$ at most once.

The image of r in the quotient $\beta(\tilde{X})$ therefore terminates in a point of $\hat{\mathcal{P}}$ when it has finite length, and a point in $\partial\beta(\tilde{X}) \setminus \hat{\mathcal{P}}$ otherwise. The same is true for both ends of a complete q -geodesic line l , and we note that both ends of l cannot land on the same point because then we would have a sequence of segments $l_n \subset l$ of length going to ∞ with both endpoints of l_n on the same edge of ∂D_n , a contradiction to the fact that l_n is a geodesic and the arcs in ∂D_n have bounded length.

Now let x, y be two distinct points in $\partial\beta(\tilde{X})$. Assume first that both are not in $\hat{\mathcal{P}}$. Then for large enough n , they are in separate components of the complement of D_n . If we let $x_i \rightarrow x$ and $y_i \rightarrow y$, then eventually x_i and y_i are in the same components of the complement of D_n as x and y , respectively. The geodesic from x_i to y_i must therefore pass through the corresponding boundary segments of D_n and in particular through D_n , so we can extract a convergent subsequence as $i \rightarrow \infty$. Letting $n \rightarrow \infty$ and diagonalizing we obtain a limiting geodesic which terminates in x, y as desired. If $x \in \hat{\mathcal{P}}$ or $y \in \hat{\mathcal{P}}$ the same argument works except that we can take $x_i \equiv x$ or $y_i \equiv y$. \square

With [Proposition 2.2](#) in hand we can consider each complete q -geodesic line l in $\hat{X} = \hat{\mathbb{H}}^2$ as an arc in the closed disk $\overline{\mathbb{H}^2}$, which by the Jordan curve theorem separates the disk \mathbb{H}^2 into at least 2 components. Each component is an open disk meeting $\partial\mathbb{H}^2$ in (a subarc of) one of the complementary arcs of the endpoints of l .

We call the union of disks meeting one of these complementary arcs an *open side* of l . The closure of each open side is then a connected union of closed disks, attached to each other along the points of $\hat{\mathcal{P}}$ that l meets on the circle. We call these closures the two *sides* of l in $\overline{\mathbb{H}^2}$. With this picture we can state the following:

Corollary 2.4. *Let a, b be disjoint arcs in \mathbb{H}^2 with well-defined, distinct endpoints on $\partial\mathbb{H}^2$ and let a_q, b_q be q -geodesic lines with the same endpoints as a and b , respectively. Then b_q is contained in a single side of a_q .*

Proof. Letting L and R be the arcs of $\partial\mathbb{H}^2$ minus the endpoints of a , the endpoints of b must lie in one of them, say L , since a and b are disjoint.

Since a_q and b_q are geodesics in the CAT(0) space \tilde{X} , their intersection is connected. If their intersection is empty, then the corollary is clear. Otherwise, $b_q \setminus a_q$ is one or two arcs, each with one endpoint on a_q and the other on L . It follows that $b_q \setminus a_q$ is on one open side of a_q , and the corollary follows. \square

Subsurfaces and projections in the flat metric. Let $Y \subset X$ be an essential compact subsurface, and let $X_Y = \tilde{X}/\pi_1(Y)$ be the associated cover of X . For any lamination λ in X , we want to show that the projection $\pi_Y(\lambda)$ can be represented by the q -geodesic representatives of λ .

We say a boundary component of Y is *puncture-parallel* if it bounds a disk in $\bar{X} \setminus Y$ that contains a single point of \mathcal{P} . We denote the corresponding subset of \mathcal{P} by \mathcal{P}_Y and refer to them as the *punctures* of Y . Let $\tilde{\mathcal{P}}_Y$ denote the subset of punctures of X_Y which are encircled by the boundary components of the lift of Y to X_Y . In terms of the completed space \bar{X}_Y , $\tilde{\mathcal{P}}_Y$ is exactly the set of completion points which have finite total angle. Let $\partial_0 Y$ denote the puncture-parallel components of ∂Y and let $\partial' Y$ denote the rest. Set $Y' = Y \setminus \partial_0 Y$.

Identifying \tilde{X} with \mathbb{H}^2 , let $\Lambda \subset \partial\mathbb{H}^2$ be the limit set of $\pi_1(Y)$, $\Omega = \partial\mathbb{H}^2 \setminus \Lambda$, and $\hat{\mathcal{P}}_Y \subset \partial\mathbb{H}^2$ the set of parabolic fixed points of $\pi_1(Y)$. Let $C(X_Y)$ denote the compactification of X_Y given by $(\mathbb{H}^2 \cup \Omega \cup \hat{\mathcal{P}}_Y)/\pi_1(Y)$, adding a point for each puncture-parallel end of X_Y , and a circle for each of the other ends. Now given a lamination (or foliation) λ , realized geodesically in the hyperbolic metric on X , its lift to X_Y extends to properly embedded arcs in $C(X_Y)$, of which the ones that are essential give $\pi_Y(\lambda)$.

Proposition 2.2 allows us to perform the same construction with the q -geodesic representative of λ . Note that the leaves we obtain may meet the boundary of $C(X_Y)$ in their interior, but a slight perturbation produces properly embedded lines in X_Y which are properly isotopic to the leaves coming from λ .

If Y is an annulus the same construction works, with the observation that the ends of Y cannot be puncture-parallel and hence $C(Y)$ is an annulus and the leaves have well-defined endpoints in its boundary. We have proved:

Lemma 2.5. *Let $Y \subset X$ be an essential subsurface. If λ is a proper arc or lamination in X , the lifts of its q -geodesic representatives to X_Y give representatives of $\pi_Y(\lambda)$.*

q -convex hulls. We will need a flat-geometry analogue of the hyperbolic convex hull for the cover X_Y . The main idea is simple – pull the regular convex hull tight using q -geodesics. The only difficulty comes from the fact that these geodesics can pass through parabolic fixed points, and fail to be disjoint from each other, so the resulting object may fail to be an embedded surface. Our discussion is similar to Section 3 of Rafi [Raf05], but the discussion there requires adjustments to handle correctly the incompleteness at punctures.

Let $\hat{i} : Y \rightarrow X_Y$ be the lift of the inclusion map to the cover.

Lemma 2.6. *The lift $\hat{i} : Y \rightarrow X_Y$ is homotopic to a map $\hat{i}_q : Y \rightarrow \bar{X}_Y$ such that*

- (1) *The homotopy $(h_t)_{t \in [0,1]}$ from \hat{i} to \hat{i}_q has the property that $h_t(Y) \subset X_Y$ for all $t \in [0, 1)$.*
- (2) *Each component of $\partial_0 Y$ is taken by \hat{i}_q to the corresponding completion point of $\bar{\mathcal{P}}_Y$.*
- (3) *If Y is an annulus then the image of \hat{i}_q is either a maximal flat cylinder in \bar{X}_Y or the unique geodesic representative of the core of Y in \bar{X}_Y .*
- (4) *If Y is not an annulus then each component γ of $\partial' Y$ is taken to a q -geodesic representative in \bar{X}_Y . If there is a flat cylinder in the homotopy class of γ then the interior of the cylinder is disjoint from $\hat{i}_q(Y)$.*
- (5) *The complementary regions of $\hat{i}_q(Y)$ in \bar{X}_Y are disks and annuli.*

Proof. As above, identify \tilde{X} with \mathbb{H}^2 , let $\Gamma = \pi_1 Y$ and let $\Lambda \subset \partial \mathbb{H}^2$ denote the limit set of Γ . Let $CH(\Lambda)$ be the convex hull of Λ in \mathbb{H}^2 , and as usual $CH(\Lambda)/\Gamma$ can be identified with Y' . After isotopy we may assume $\hat{i} : Y' \rightarrow CH(\Lambda)/\Gamma$ is this identification.

First assume that Y is not an annulus. Using [Proposition 2.2](#) we can mimic this hull construction in the q metric. Each boundary geodesic l of $CH(\Lambda)$ has the same endpoints as a (biinfinite) q -geodesic l_q , whose quotient is a geodesic representative of a component of ∂Y . The q -geodesic may pass through points of $\hat{\mathcal{P}}$, so that the homotopy between l and l_q can be chosen to stay in \mathbb{H}^2 until the last instant. Note also that l_q is unique unless it is part of a parallel family of geodesics, whose quotient in X_Y is a flat annulus.

The plane is divided by l_q into two sides as in the discussion before [Corollary 2.4](#), and one of the sides, which we call Ω_l , meets S^1 in the complement of Λ . Recall that Ω_l is either a half-plane or a string of disks attached along puncture points. The quotient in X_Y is therefore either an annulus or a union of disks attached along punctures. If l_q is one of a parallel family of geodesics, we include this family in Ω_l . After deleting from \hat{X} the interiors of Ω_l for all l in $\partial CH(\Lambda)$ (which are disjoint by [Corollary 2.4](#)), we obtain $CH_q(\Lambda)$, the q -convex hull. We may equivariantly deform the identity to a map $CH(\Lambda) \rightarrow CH_q(\Lambda)$, which takes each l to l_q : since $CH_q(\Lambda)$ is contractible, it suffices to give a Γ -invariant triangulation of $CH(\Lambda)$ and define the homotopy successively on the skeleta. This homotopy descends to a map from Y' to $CH_q(\Lambda)/\Gamma$, and can be chosen so that the puncture-parallel ends map to the corresponding points of \mathcal{P}_Y . This gives the desired map \hat{i}_q .

When Y is an annulus, we let $CH_q(\Lambda)$ be the q geodesics joining the two points of Λ . This is either a flat strip in \hat{X} which descends to a flat cylinder in \bar{X}_Y , or it is a single geodesic. The proof in this case now proceeds exactly as above. \square

Let $\iota_q : Y \rightarrow \bar{X}$ be the composition of $\hat{\iota}_q$ with the (branched) covering $\bar{X}_Y \rightarrow \bar{X}$ and set $\partial_q Y = \iota_q(\partial' Y)$. Note that this will be a 1-complex of saddle connections and not necessarily a homeomorphic image of $\partial' Y$.

2.4. Fibered faces of the Thurston norm. A fibration $\sigma : M \rightarrow S^1$ of a finite-volume hyperbolic 3-manifold M over the circle comes with the following structure: there is an integral cohomology class in $H^1(M; \mathbb{Z})$ represented by $\sigma_* : \pi_1 M \rightarrow \mathbb{Z}$, which is the Poincaré dual of the fiber F . There is a representation of M as a quotient $F \times \mathbb{R}/\Phi$ where $\Phi(x, t) = (f(x), t - 1)$ and $f : F \rightarrow F$ is called the monodromy map. This map is pseudo-Anosov and has stable and unstable (singular) measured foliations λ^+ and λ^- on F . Finally there is the suspension flow inherited from the natural \mathbb{R} action on $F \times \mathbb{R}$, and suspensions Λ^\pm of λ^\pm which are flow-invariant 2-dimensional foliations of M . All these objects are defined up to isotopy.

The fibrations of M are organized by the *Thurston norm* $\|\cdot\|$ on $H^1(M; \mathbb{R})$ [Thu86]. (See also [CC00].) This norm has a polyhedral unit ball B with the following properties:

- (1) Every cohomology class dual to a fiber is in the cone $\mathbb{R}_+ \mathcal{F}$ over a top-dimensional face \mathcal{F} of B .
- (2) If $\mathbb{R}_+ \mathcal{F}$ contains a cohomology class dual to a fiber then *every* integral class in $\mathbb{R}_+ \mathcal{F}$ is dual to a fiber. \mathcal{F} is called a *fibered face* and its integral classes are called fibered classes.
- (3) For a fibered class ω with associated fiber F , $\|\omega\| = -\chi(F)$.

In particular if $\dim H^1(M; \mathbb{R}) \geq 2$ and M is fibered then there are infinitely many fibrations, with fibers of arbitrarily large complexity. We will abuse terminology a bit by saying that a fiber (rather than its Poincaré dual) is in $\mathbb{R}_+ \mathcal{F}$.

The fibered faces also organize the suspension flows and the stable/unstable foliations: If \mathcal{F} is a fibered face then there is a single flow ψ and a single pair Λ^\pm of foliations whose leaves are invariant by ψ , such that *every* fibration associated to $\mathbb{R}_+ \mathcal{F}$ may be isotoped so that its suspension flow is ψ , and the foliations λ^\pm for the monodromy of its fiber F are $\Lambda^\pm \cap F$. These results were proven by Fried [Fri82]; see also McMullen [McM00].

Veering triangulation of a fibered face. A key fact for us is that the veering triangulation of the manifold M depends only on the fibered face \mathcal{F} and not on a particular fiber. This was known to Agol for his original construction (see sketch in [Ago12]), but Guéritaud's construction makes it almost immediate.

Proposition 2.7 (Invariance of τ). *Let S_1 and S_2 be fibers of M each contained in $\mathbb{R}_+ \mathcal{F}$ and let τ_1 and τ_2 be the corresponding veering triangulations of M . Then, after an isotopy preserving transversality to the suspension flow, $\tau_1 = \tau_2$.*

Proof. The suspension flow associated to \mathcal{F} lifts to the universal cover \widetilde{M} , and any fiber S in $\mathbb{R}_+\mathcal{F}$ is covered by a copy of its universal cover \widetilde{S} in \widetilde{M} which meets every flow line transversely, exactly once. Thus we may identify \widetilde{S} with the leaf space \mathcal{L} of this flow. The lifts $\widetilde{\Lambda}^\pm$ of the suspended laminations project to the leaf space where they are identified with the lifts $\widetilde{\lambda}^\pm$ of λ^\pm to \widetilde{S} .

The foliated rectangles used in the construction of τ from \tilde{q} on \widetilde{S} depend only on the (unmeasured) foliations $\widetilde{\lambda}^\pm$. Thus the abstract cell structure of τ depends only on the fibered face \mathcal{F} and not on the fiber. The map π from each tetrahedron to its rectangle does depend a bit on the fiber, as we choose q -geodesics for the edges (and the metric q depends on the fiber); but the edges are always mapped to arcs in the rectangle that are transverse to both foliations. It follows that there is a transversality-preserving isotopy between the triangulations associated to any two fibers. \square

Fibers and projections. We next turn to a few lemmas relating subsurface projections over the various fibers in a fixed face of the Thurston norm ball.

Lemma 2.8. *If \mathcal{F} is a fibered face for M and $Y \rightarrow S$ is an infinite covering where S is a fiber in $\mathbb{R}_+\mathcal{F}$ and $\pi_1(Y)$ is finitely-generated, then the projection $d_Y(\lambda^-, \lambda^+)$ depends only on \mathcal{F} and the conjugacy class of the subgroup $\pi_1(Y) \leq \pi_1(M)$ (and not on S).*

Proof. As in the proof of [Proposition 2.7](#), \widetilde{S} can be identified with the leaf space \mathcal{L} of the flow in \widetilde{M} . The action of $\pi_1(M)$ on \widetilde{M} descends to \mathcal{L} , and thus the cover $Y = \widetilde{S}/\pi_1(Y)$ is identified with the quotient $\mathcal{L}/\pi_1(Y)$ and the lifts of λ^\pm to Y are identified with the images of $\widetilde{\Lambda}^\pm$ in $\mathcal{L}/\pi_1(Y)$. Thus the projection $d_Y(\lambda^+, \lambda^-)$ can be obtained without reference to the fiber S . \square

This lemma justifies the notation $d_Y(\Lambda^+, \Lambda^-)$ used in the introduction.

We will also require the following lemma, where we allow maps homotopic to fibers which are not necessarily embeddings.

Lemma 2.9. *Let F be a fiber of M . Let $Y \subset M$ be a compact surface and let $h: F \rightarrow M$ be a map which is homotopic to the inclusion. Suppose that $h(F) \cap Y$ is inessential in Y , i.e. each component of the intersection is homotopic into the ends of Y . Then the image of $\pi_1(Y)$ is contained in $\pi_1(F)$.*

Proof. Let ζ be the cohomology class dual to F . Since $h(F)$ meets Y inessentially, every loop in Y can be pushed off of $h(F)$ so ζ vanishes on $\pi_1(Y)$. But the kernel of ζ in $\pi_1(M)$ is exactly $\pi_1(F)$, so the image of $\pi_1(Y)$ is in $\pi_1(F)$. \square

3. SECTIONS AND POCKETS OF THE VEERING TRIANGULATION

A *section* of the veering triangulation τ is an embedding $(X, T) \rightarrow (X \times \mathbb{R}, \tau)$ which is simplicial with respect to an ideal triangulation T of X , and is a section of the fibration π (hence transverse to the vertical flow). The edges of T are saddle connections of q that are also edges of τ (i.e. those which span singularity-free

rectangles), and indeed any triangulation by τ -edges gives rise to a section. We will abuse terminology a bit by letting T denote both the triangulation and the section.

A *diagonal flip* $T \rightarrow T'$ between sections is an isotopy that pushes T through a single tetrahedron of τ , either above it or below it. Equivalently, if R is a maximal rectangle and Q its associated tetrahedron, the bottom two faces of Q might appear in T , in which case T' would be obtained by replacing these with the top two faces. This is an upward flip, and the opposite is a downward flip. We will refer to the transition as both a *diagonal flip/exchange* and a *tetrahedron move*, depending on the perspective.

An edge e of T can be flipped downward exactly when it is the tallest edge, with respect to q , among the edges in either of the two triangles adjacent to it. This makes e the top edge of a tetrahedron (i.e. the diagonal of a quadrilateral that connects the horizontal of the corresponding rectangle). Similarly it can be flipped upward when it is the widest edge among its neighbors. See [Figure 5](#).

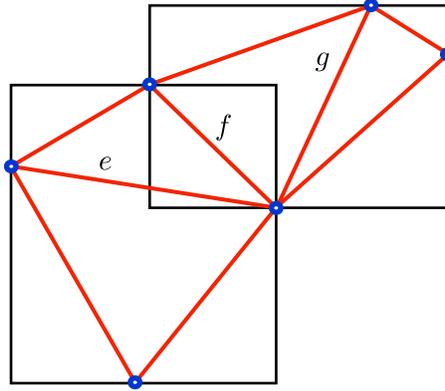


FIGURE 5. The edge e is upward flippable, g is downward flippable, and f is not flippable.

In particular it follows that every section has to admit both an upward and downward flip – simply find the tallest edge and the widest edge.

However it is not a priori obvious that a section even exists. Guéritaud gives an argument for this and more:

Lemma 3.1 ([Gué15]). *There is a sequence of sections $\cdots \rightarrow T_i \rightarrow T_{i+1} \rightarrow \cdots$ separated by upward diagonal flips, which sweeps through the entire manifold $(X \times \mathbb{R}, \tau)$. Moreover, when $(X \times \mathbb{R}, \tau)$ covers the manifold (M, τ) , this sequence is invariant by the deck translation Φ .*

For an alternative proof that sections exist, see the second proof of [Lemma 3.2](#). We remark that [Lemma 3.1](#) does not give a complete picture of all possible sections of τ . In this section we will establish a bit more structure.

For a subcomplex $K \leq \tau$, denote by $T(K)$ the collection of sections T of τ containing the edges of K . A necessary condition for $T(K)$ to be nonempty is that

$\pi(K)$ is an embedded complex in X composed of τ simplices. We will continue to blur the distinction between K and $\pi(K)$.

Our first result states that any embedded K can be extended to a section:

Lemma 3.2 (Extension lemma). *Suppose that E is a collection of τ -edges in X with pairwise disjoint interiors. Then $T(E)$ is nonempty.*

The second states that $T(K)$ is always connected by tetrahedron moves. This includes in particular the case of $T(\emptyset)$, the set of all sections.

Proposition 3.3 (Connectivity). *If K is an embedded subcomplex of τ , then $T(K)$ is connected via tetrahedron moves.*

Finding flippable edges. Let T be a section and let σ be an edge of τ , which is not an edge of T . Any edge e of T crossing σ must do so from top to bottom ($e > \sigma$) or left to right ($e < \sigma$), as in [Section 2.1](#), and we further note that all edges of T that cross σ do it consistently, all top-bottom or all left-right, since they are disjoint from each other.

Lemma 3.4. *Let T be a section and suppose that an edge σ of τ is crossed by an edge e of T with $e > \sigma$. Then there is an edge of T crossing σ which is downward flippable. Similarly if $e < \sigma$ then there is an edge of T crossing σ which is upward flippable.*

Proof. Assuming the crossings of σ are top to bottom, let e be the edge crossing σ that has largest height with respect to q . Let D be a triangle of T on either side of e . Then e must be the tallest edge of D . If not then let f be the tallest edge. Drawing the rectangle M in which D is inscribed ([Figure 6](#)) one sees that R , the rectangle of σ , is forced to cross it from left to right. Hence the edge f must also cross σ . This contradicts the choice of e . It follows that e is a downward flippable edge. \square

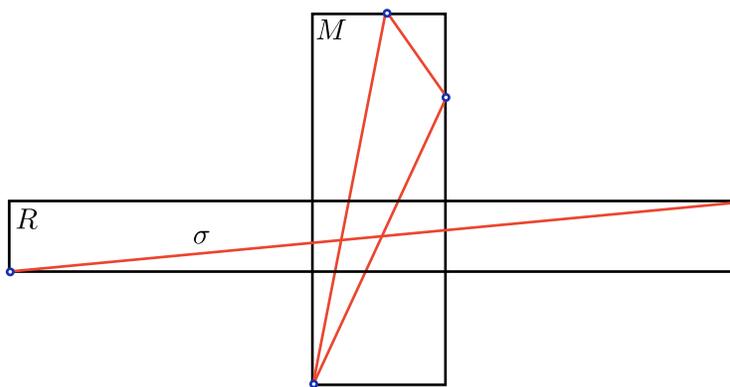


FIGURE 6. The tallest T -edge crossing σ must also be tallest in its own triangles.

Pockets. Let T and T' be two sections and K their intersection. Because both sections are transverse to the suspension flow, their union $T \cup T'$ divides $X \times \mathbb{R}$ into two unbounded regions and some number of bounded regions. Each bounded region U is a union of tetrahedra bounded by two isotopic subsurfaces of T and T' , which correspond to a component W of the complement of $\pi(K)$ in X . The isotopy is obtained by following the flow, and if it takes the subsurface of T' upward to the subsurface of T we say that T lies above T' in U . We call U a *pocket over W* , and sometimes write U_W .

Lemma 3.5. *With notation as above, T lies above T' in the pocket U_W if and only if, for every edge e of T in W and edge e' of T' in W , if e and e' cross then $e > e'$.*

Proof. Let e be an edge of T in W ; hence, it is in the top boundary of U . Let Q be the tetrahedron of τ for which e is the top edge. Via the local picture around e we see that Q lies locally below T . Its interior is of course disjoint from T and T' (and the whole 2-skeleton), hence it is inside U . Let e_1 be the bottom edge of Q (so that replacing e with e_1 is a downward diagonal flip). Note $e > e_1$. If e_1 is in T' , stop. Otherwise it is in the interior of U , and we can repeat with the tetrahedron for which e_1 is the top edge. We get a sequence of moves terminating in some e' in T' , which must be in the boundary of U , and conclude $e > e'$ (by the transitivity of $>$ as in [Section 2.1](#)). Now from the previous discussion the same slope relation holds for every edge of T' crossing e , and for every edge of T crossing e' . Arguing in the opposite direction we can start with an edge of T' and find the corresponding edges of T . The lemma follows. \square

Connectedness of $T(K)$. We can now prove [Proposition 3.3](#).

Proof. Let us consider T, T' in $T(K)$. Let U be one of the pockets, and suppose T lies above T' in U . [Lemma 3.5](#) together with [Lemma 3.4](#) implies that T has a downward flippable edge in the pocket, and T' has an upward flippable edge. Performing either one of these flips we reduce the size of this pocket. Thus a finite number of moves will take T to T' or vice versa, without disturbing K . \square

Corollary 3.6. *If K is a subcomplex of τ and $T(K) \neq \emptyset$, then there are unique sections $T^+(K)$ and $T^-(K)$ in $T(K)$ such that every $T \in T(K)$ can be up flipped to $T^+(K)$ and down flipped to $T^-(K)$.*

The section $T^+(K)$ is called the *top of $T(K)$* and the section $T^-(K)$ is called the *bottom of $T(K)$* . Note that any section obtained from $T^+(K)$ by upward diagonal exchanges is not in $T(K)$.

Extension lemma. We conclude this section with two proofs of [Lemma 3.2](#).

Proof one. [Lemma 3.1](#) gives us, in particular, the existence of at least one section T_0 which is disjoint from E , which we may assume lies above every edge of E .

Then by [Lemma 3.4](#) there is a downward flippable edge e in T_0 . The tetrahedron involved in the move lies above E , so E still lies below (or is contained in) the new section T_1 . We repeat this process, and at each stage every edge of E is either

contained in T_i or lies below it, and each T_i contains a downward flippable edge that is not contained in $E \cap T_i$.

Because τ is locally finite at each edge, *any* sequence of downward flips must eventually meet every edge of τ below T_0 . Thus we may continue until every edge of E lies in T_i . \square

Proof two. Our second proof does not use [Lemma 3.1](#), and in particular it gives an independent proof of the existence of sections.

Let D be a component of the complement of E which is not a triangle. Let e be an edge of ∂D and consider the collection of τ -tetrahedra adjacent to e . These contain a sequence $Q_-, Q_1, \dots, Q_m, Q_+$, as in [Figure 4](#), where Q_- is the tetrahedron with e as its top edge, Q_+ is the tetrahedron with e as its bottom edge, and the rest are adjacent to e on the same side as D (if D meets e on two sides we just choose one). Two successive tetrahedra in this sequence share a triangular face. We claim that one of these faces must be contained in D . Equivalently we claim that one of the triangles is not crossed by any edge of E .

Since each tetrahedron Q is inscribed in a singularity free rectangle R , if an edge f of E crosses any edge of Q its rectangle crosses all of R . It follows immediately, since the edges of E have disjoint interiors, that they consistently cross R all vertically, or all horizontally. Because successive tetrahedra in the sequence share a face it follows inductively that, if all the faces are crossed by E , then they are all consistently crossed horizontally, or all vertically.

However, Q_- can only be crossed vertically by E (since E does not cross e). Similarly Q_+ can only be crossed horizontally. It follows that there must be a triangular face F that is *not* crossed by E . Thus F is contained in D . Since D is not a triangle, at least one edge of F passes through the interior of D . We add this edge to E and proceed inductively. \square

4. RECTANGLE AND TRIANGLE HULLS

In this section we discuss a number of constructions that associate a configuration of τ -edges to a saddle connection of the quadratic differential q . These will be used later to show that subsurfaces with large projection are compatible with the veering triangulation in the appropriate sense. As a byproduct of our investigation, we prove the (to us) unexpected result ([Theorem 1.4](#)) that the edges of the veering triangulation form a totally geodesic subgraph of the curve and arc graph of X .

4.1. Maximal rectangles along a saddle connection. Let σ be a saddle connection, for the moment in the completed universal cover \hat{X} . Consider the set $\mathcal{R}(\sigma)$ of all rectangles which are *maximal with respect to the property that σ passes through a diagonal*. Thus each $R \in \mathcal{R}(\sigma)$ contains singularities in at least two edges. Let $h(R)$ be the convex hull in R of the singularities in its boundary (see [Figure 7](#)).

Let

$$\mathbf{r}(\sigma) = \bigcup \{ \partial h(R) : R \in \mathcal{R}(\sigma) \}.$$

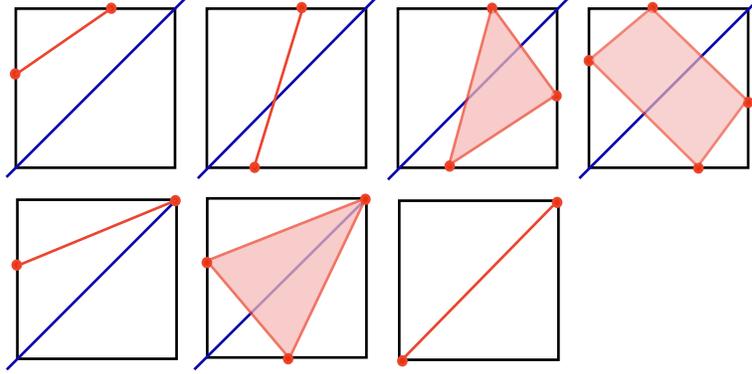


FIGURE 7. The seven possible convex hulls $h(R)$, assuming at most one singularity per leaf of λ^\pm . The saddle connection σ is in blue.

See [Figure 8](#) for an example. Note that all the saddle connections in $\mathbf{r}(\sigma)$ are edges of τ — each of these arcs spans a singularity-free rectangle by construction. Moreover, $\mathbf{r}(\sigma) = \sigma$ if σ is itself a τ edge.

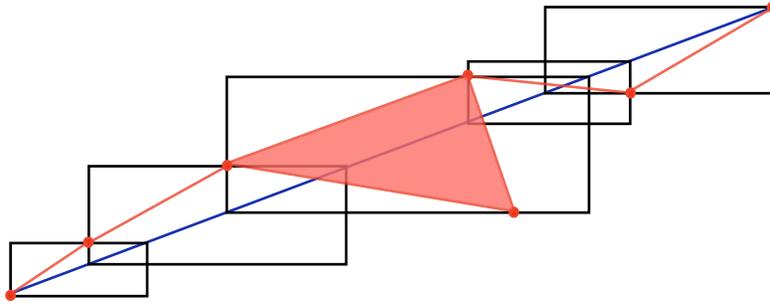


FIGURE 8. Example of $\mathbf{r}(\sigma)$ (in red)

The following lemma will play an important role throughout this paper.

Lemma 4.1. *If σ_1 and σ_2 have no transversal intersections then neither do $\mathbf{r}(\sigma_1)$ and $\mathbf{r}(\sigma_2)$.*

Proof. Say that two rectangles meet *crosswise* if their interiors intersect, and no corners of one are in the interior of the other. Note that when two rectangles meet crosswise, any two of their diagonals either intersect or are equal.

Suppose that $R_1 \in \mathcal{R}(\sigma_1)$ and $R_2 \in \mathcal{R}(\sigma_1)$ have intersecting interiors, but do not meet crosswise. Since there are no singularities in their interiors, the singularities of R_1 must be in $\partial R_1 \setminus R_2$ and vice-versa for R_2 . This means that the convex hulls $h(R_i)$ must lie in the hulls of $\partial R_i \setminus R_{2-i}$, and hence have disjoint interiors, as in [Figure 9](#). It follows that $\partial h(R_1)$ and $\partial h(R_2)$ cannot cross. Conversely, if they do cross then the rectangles meet crosswise and since σ_1 and σ_2 pass through their diagonals, either they cross or they are the same saddle connection and $R_1 = R_2$. \square

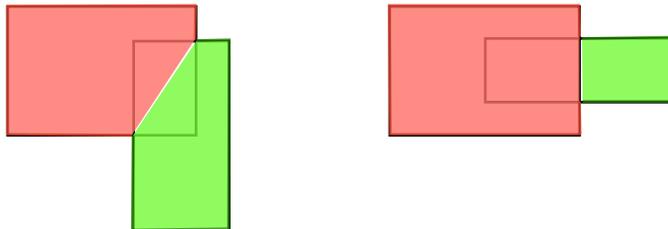


FIGURE 9. The hull of $\partial R_1 \setminus R_2$ is in red and the hull of $\partial R_2 \setminus R_1$ is in green.

The proof of the preceding lemma uses a convex hull argument similar to that of Guéritaud, used in his proof of [Lemma 3.1](#) [[Gué15](#), Proposition 2.1]. There, it is applied to squares; using maximal squares in his setting to build a section of τ is the same as, in our setting, considering simultaneously all saddle connections (and complete leaves) of slope 1.

An immediate consequence of [Lemma 4.1](#) is that we can carry on the construction downstairs: If σ is a saddle connection in \bar{X} we can construct $\mathbf{r}(\hat{\sigma})$ for each of its lifts $\hat{\sigma}$ to \hat{X} , and the lemma tells us none of them intersect transversally. Thus the construction projects downstairs to give an embedded subcomplex. Moreover if K is *any* disjoint collection of saddle connections then $\mathbf{r}(K)$ makes sense as an embedded subcomplex of τ . Hence, we will continue to use $\mathbf{r}(\cdot)$ to denote the corresponding map on saddle connections of \bar{X} .

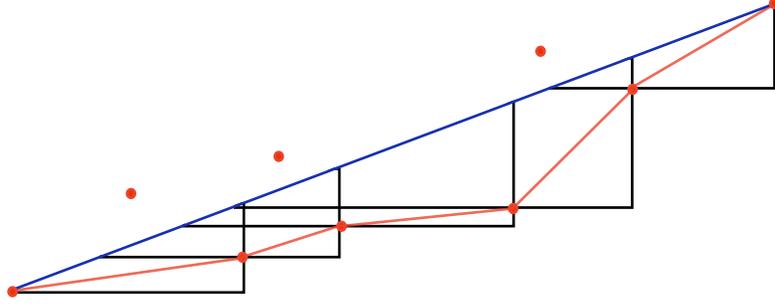
4.2. Triangle hulls. Now let us consider a similar operation that uses right triangles instead of rectangles, and associates to a transversely oriented saddle connection in the universal cover a homotopic path of saddle connections.

If σ is a saddle connection in \hat{X} equipped with a transverse orientation, let $\mathcal{T}(\sigma)$ denote the collection of Euclidean right triangles which are *maximal with respect to the property that they are attached along the hypotenuse to σ along the side given by its transverse orientation*. A triangle t in $\mathcal{T}(\sigma)$ must have exactly one singularity in each of its legs, and so their convex hull $h(t)$ is a single saddle connection. The set $\mathcal{T}(\sigma)$ must be finite, and its hypotenuses cover σ in a sequence of non-nested intervals. See [Figure 10](#). Let $\mathbf{t}(\sigma)$ be the union of segments $h(t)$ for $t \in \mathcal{T}(\sigma)$.

Lemma 4.2. *Either $\mathbf{t}(\sigma) = \sigma$ or $\sigma \cup \mathbf{t}(\sigma)$ is the boundary of an embedded Euclidean polygon in \hat{X} .*

Proof. Suppose that t and t' are triangles of $\mathcal{T}(\sigma)$ and $p \in t \cap t'$ is in the interior of t . Let l and l' be the vertical line segments in t and t' , respectively, joining p to the respective hypotenuses (l' could be a single point). If l and l' leave p in opposite directions then $l \cup l'$ is a vertical geodesic connecting two points of σ , which contradicts the uniqueness of geodesics in \hat{X} . If they leave p in the same direction but are not equal, then their difference is a vertical geodesic with endpoints on σ , again a contradiction.

We conclude that if t and t' intersect they do so on a common subarc of their hypotenuses. This subarc spans a (nonmaximal) right triangle which is exactly $t \cap t'$.

FIGURE 10. An example of $\mathbf{t}(\sigma)$.

Now given $t \in \mathcal{T}(\sigma)$, the vertical and horizontal legs of t each contain a single singularity of \hat{X} ; denote these singularities by v_t and h_t , respectively. By construction, there is a unique triangle $t' \in \mathcal{T}(\sigma)$ such that $h_{t'} = v_t$, unless v_t is an endpoint of σ . Hence, given an orientation on σ , the edges of $\mathbf{t}(\sigma)$ come with a natural ordering induced by moving along σ . By our observations above, we see that $\mathbf{t}(\sigma)$ is an embedded arc and meets σ only at its endpoints. Since \hat{X} is contractible σ and $\mathbf{t}(\sigma)$ must be homotopic and hence cobound a disk. In fact this disk is foliated by both λ^+ and λ^- , as we can see by noting that each edge of $\mathbf{t}(\sigma)$ cobounds a vertical (similarly a horizontal) strip with a segment in σ . Hence this disk admits an isometry to a polygon in \mathbb{R}^2 . \square

Unlike the rectangle hulls, the edges of $\mathbf{t}(\sigma)$ are not necessarily τ -edges. Moreover, the \mathbf{t} -version of Lemma 4.1 is in general not true. That is, \mathbf{t} may not project to an embedded complex in \bar{X} since σ_1 and σ_2 can be disjoint while $\mathbf{t}(\sigma_1)$ and $\mathbf{t}(\sigma_2)$ cross.

4.3. Retractions in \mathcal{A} . In this subsection, X is fully-punctured. Let $\mathcal{A}(\tau) \subset \mathcal{A}(X)$ be the span of the vertices of $\mathcal{A}(X)$ which are represented by edges of τ . We will construct a 1-Lipschitz retraction from $\mathcal{A}(X)$ to $\mathcal{A}(\tau)$. First, let $\mathcal{SC}(q) \subset \mathcal{A}(X)$ be the arcs of X which can be realized by saddle connections of q . For any $a \in \mathcal{A}(X)$ define $\mathbf{s}(a) \subset \mathcal{SC}(q)$ as follows: If a_q is the q -geodesic representative of a in \bar{X} , then let $\mathbf{s}(a)$ be the set of saddle connections of q composing a_q . If a is a cylinder curve of q , then we take $\mathbf{s}(a)$ to be the set of saddle connections appearing in the boundary of the maximal cylinder of a . Note that if $a \in \mathcal{A}(X)$ is itself represented by a saddle connection of q , then $\mathbf{s}(a) = a$.

The following lemma shows that \mathbf{s} is well-defined and “1-Lipschitz”, in the sense that it takes diameter 1 sets to diameter 1 sets.

Lemma 4.3. *For adjacent vertices $a, b \in \mathcal{A}(X)$, the vertices of $\mathbf{s}(a)$ and $\mathbf{s}(b)$ are pairwise adjacent.*

Proof. Recall that adjacency of vertices in $\mathcal{A}(X)$ corresponds to disjointness of their hyperbolic geodesic representative, and for vertices realized by saddle connections, this corresponds to the lack of transverse intersection of their interiors. But if any

arcs of $\mathbf{s}(a)$ and $\mathbf{s}(b)$ have crossing interiors, [Corollary 2.4](#) implies that the hyperbolic geodesic representatives of a and b must cross as well. The lemma follows. \square

Combining this lemma with [Lemma 4.1](#) gives us the proof of [Theorem 1.4](#), which we restate here in somewhat more precise language:

Theorem 1.4 (Totally geodesic theorem). *Let (X, q) be fully punctured with veering triangulation τ . The composition $\mathbf{r} \circ \mathbf{s} : \mathcal{A}(X) \rightarrow \mathcal{A}(\tau)$ is 1-Lipschitz retraction in the sense that it takes diameter 1 sets to diameter 1 sets, and is the identity on $\mathcal{A}(\tau)$. Hence, any two vertices in $\mathcal{A}(\tau)$ are joined by a geodesic of $\mathcal{A}(X)$ that lies in $\mathcal{A}(\tau)$.*

Proof. [Lemma 4.3](#) says that $\mathbf{s} : \mathcal{A}(X) \rightarrow \mathcal{SC}(q)$ is a 1-Lipschitz retraction. [Lemma 4.1](#), interpreted as a statement about the arc and curve complexes, says the same for $\mathbf{r} : \mathcal{SC}(q) \rightarrow \mathcal{A}(\tau)$. The theorem follows. \square

5. PROJECTIONS AND COMPATIBLE SUBSURFACES

In this section we show that if $Y \subset X$ is a compact essential subsurface of large projection $d_Y(\lambda^+, \lambda^-)$, then Y has a particularly nice representation with respect to, first, the quadratic differential q and, second, the veering triangulation τ . We emphasize that in this section, the surface X is not necessarily fully-punctured. Thus by τ we mean the veering triangulation associated to the fully-punctured surface $X \setminus \text{sing}(q)$. We will say that a saddle connection of X is a τ edge if its interior is an edge of this veering triangulation. In particular this means that its lift to \tilde{X} spans a singularity-free rectangle.

5.1. Projection and q -compatibility. Recall the q -convex hull map $\hat{\iota}_q : Y \rightarrow \bar{X}_Y$ constructed in [Lemma 2.6](#). We say that Y is q -compatible if

- (1) $\hat{\iota}_q$ can be chosen an embedding except on the puncture-parallel boundary components $\partial_0 Y$,
- (2) its projection ι_q to \bar{X} is an embedding on $\text{int}(Y)$, and
- (3) the geodesic representatives of $\partial' Y$ do not pass through the points of $\tilde{\mathcal{P}}_Y$.

Note that Y is a q -compatible annulus if and only if the core of Y is a cylinder curve in X . In this case, the corresponding open flat cylinder in X is $\iota_q(\text{int}(Y))$. In general, if Y is q -compatible then a component of $X \setminus \partial_q Y$ is an open subsurface isotopic to the interior of Y ; this is the image $\iota_q(\text{int}(Y))$ and is denoted $\text{int}_q(Y)$.

The following proposition shows that mild assumptions on $d_Y(\lambda^+, \lambda^-)$ imply that Y is q -compatible.

Proposition 5.1 (q -Compatibility). *Let $Y \subset X$ be a non-annular essential subsurface. If $d_Y(\lambda^+, \lambda^-) > 0$, then Y is q -compatible.*

If Y is an annulus and $d_Y(\lambda^+, \lambda^-) > 1$, then Y is q -compatible. In this case, $\text{int}_q(Y)$ is a flat cylinder.

Proof. We treat the non-annular case first.

If $p \in \tilde{\mathcal{P}}_Y$ is a puncture of Y and γ is a boundary component of $\partial'Y$, we must show that $\gamma_q = \hat{i}_q(\gamma)$ does not pass through p . If γ_q passes through p , let A_γ denote a component of $X_Y \setminus \hat{i}_q(Y)$ adjacent to γ_q and meeting p . The angle in A_γ between the incoming and outgoing edges of γ_q is at least π , which implies that A_γ contains a horizontal and a vertical ray l^-, l^+ emanating from p . (Figure 11.)

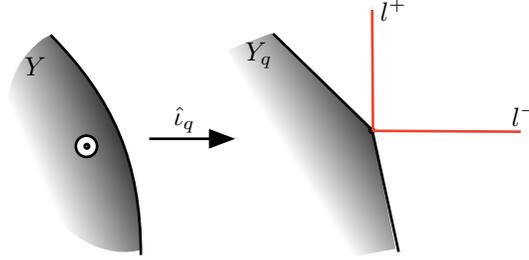


FIGURE 11. When $\hat{i}_q(\partial Y)$ passes through a point of \mathcal{P}_Y , $d_Y(\lambda^+, \lambda^-) = 0$.

These rays are proper q -geodesic lines in X_Y (because p is a puncture, not a point of X_Y), and hence by Lemma 2.5 represent vertices of $\pi_Y(\lambda^-)$ and $\pi_Y(\lambda^+)$, respectively. Further, since the rays only intersect within the annulus or disk A_γ and Y is itself nonannular, we see that l^- and l^+ in fact represent the same point in $\mathcal{A}(Y)$. (Actually, if A_γ does not contain a flat cylinder, then the interiors of l^- and l^+ are disjoint as we show below). Either way, it follows that

$$d_Y(\lambda^+, \lambda^-) = 0,$$

which contradicts our hypothesis.

Next we must show that \hat{i}_q is (homotopic to) an embedding. If this fails then there must be regions A, A' (not necessarily distinct) in the complement of $\hat{i}_q(Y)$ which touch, either along a common saddle connection or along a singularity.

If A and A' touch along a saddle connection σ , we have the situation of Figure 12. Any point of σ is crossed by a pair l^+, l^- of leaves of λ^+, λ^- , which cobound a disk in each of A and A' . Again we see that l^+ and l^- determine the same vertex of $\mathcal{A}(Y)$ and conclude once again that $d_Y(\lambda^+, \lambda^-) = 0$.

If A and A' touch along a singularity x that is not in \mathcal{P} , then as before there is an angle of at least π on each side, and we can find pairs of rays r_0^\pm emanating from x on the A side, and r_1^\pm emanating on the A' side (see Figure 13). The unions $l^+ = r_0^+ \cup_x r_1^+$ and $l^- = r_0^- \cup_x r_1^-$ again determine the same point in $\mathcal{A}(Y)$ and we conclude that $d_Y(\lambda^+, \lambda^-) = 0$.

In the last case, the singularity x is in $\mathcal{P} \setminus \mathcal{P}_Y$, and hence a point of infinite branching for $\bar{X}_Y \rightarrow \bar{X}$. Now A' , say, is in the infinitely branching side of $\hat{i}_q(Y)$, A is on the finitely branching side where there is an angle of at least π along the boundary at x (see Figure 14). A pair of rays l^\pm emanating from x into A are properly embedded

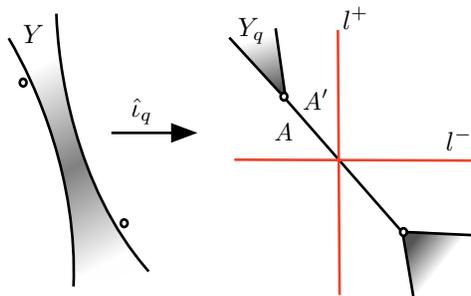


FIGURE 12. Y_q is pinched along a saddle connection.

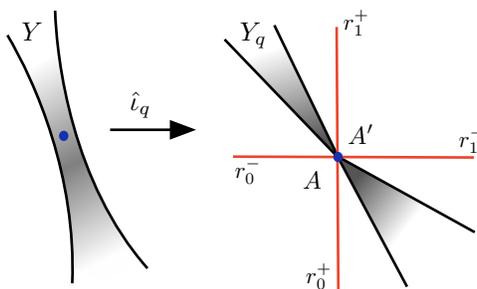


FIGURE 13. Y_q is pinched at a singularity not in \mathcal{P} .

lines and again represent the same vertex of $\mathcal{A}(Y)$, giving us $d_Y(\lambda^+, \lambda^-) = 0$ yet again.

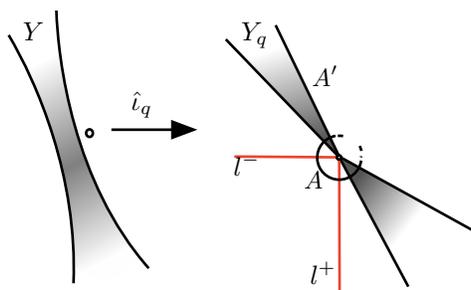


FIGURE 14. Y_q is pinched at a singularity in $\mathcal{P} \setminus \mathcal{P}_Y$.

When Y is an annulus, almost the same argument applies. The difference is that the arcs l^\pm we obtain are not homotopic with fixed endpoints, and so do not determine the same vertex of $\mathcal{A}(Y)$. However, in each case we will show they have

disjoint interiors, concluding $d_Y(l^+, l^-) \leq 1$, and so

$$d_Y(\lambda^+, \lambda^-) \leq 1.$$

To see this, let γ denote the core of Y and let γ_q be its geodesic representative in \bar{X}_Y . Supposing that $\text{int}_q(Y)$ is not a flat annulus, we first claim the following: For any singular point p crossed by γ_q , if l^+ and l^- are rays of λ^+ and λ^- , respectively, meeting with angle $\pi/2$ at p , then the interiors of l^+ and l^- do not meet.

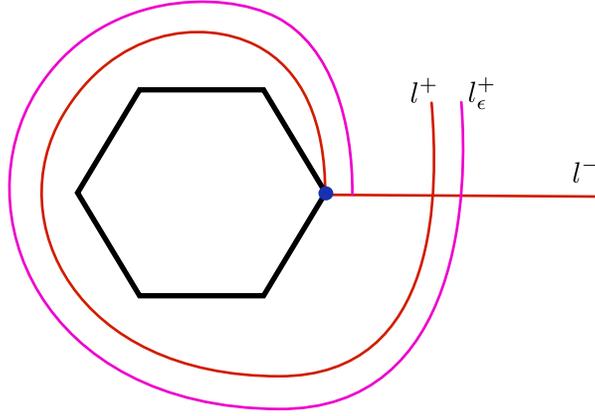


FIGURE 15. The q -geodesic γ_q is the black hexagon. An interior intersection between l^+ and l^- contradicts the Gauss–Bonnet theorem.

To establish the claim, assume that the interiors of l^\pm meet and refer to [Figure 15](#). Let A_γ be the complementary region of γ_q containing p' , the interior intersection of l^\pm . If A_γ is a disk, then the claim follows immediately from the uniqueness of geodesics in a CAT(0) space. Hence, we may assume that A_γ is an annulus. Let l_ϵ^+ be leaf of λ^+ parallel to l^+ and slightly displaced to the interior of A_γ , so that the region R bounded by γ_q and the segments of l^- and l_ϵ^+ is an annulus. The total curvature of the $l^-l_\epsilon^+$ boundary of R is 0 since it is straight except for two right turns of opposite signs, and the total curvature of γ_q as measured from inside R is nonpositive (since each singularity on γ_q subtends at least angle π within R). Since $\chi(R) = 0$ and the Gaussian curvature in R (including singularities) is nonpositive, the Gauss–Bonnet theorem implies that the total curvature of ∂R is nonnegative. This implies that the total curvature of γ_q is 0, which means that γ_q bounds a flat cylinder, which is a contradiction. This establishes the claim.

We now return to the proof of the proposition. First suppose that γ_q passes through a completion point x of \bar{X}_Y . Then, just as in [Figure 14](#), we can find a pair of rays l^\pm emanating from x into A_γ . By the claim above, the interiors of these rays do not meet and so $d_Y(\lambda^+, \lambda^-) \leq 1$ as desired.

Finally, suppose that γ_q remains in X_Y , i.e. it does not pass through any completion points. It must still pass through a singularity x , and we note that the total

angle at x is at least 3π . Recall that γ_q subtends at least angle π at x to either of its sides and we note that some side of γ sees angle at least $3\pi/2$ at x . Let A denote this side of γ_q and let A' denote the other side. Note that $A \neq A'$ since X_Y is an annulus which γ_q separates. The angle of $3\pi/2$ tells us there are at least 3 rays of λ^\pm emanating into A . Now choose rays r_0^\pm of λ^\pm emanating from x on the A' side. From the 3 rays of λ^\pm emanating from x into A , we can choose rays r_1^\pm of λ^\pm such that in the cyclic ordering of directions at x , r_0^+ is adjacent to r_1^+ and r_0^- is adjacent to r_1^- . The unions $l^+ = r_0^+ \cup_x r_1^+$ and $l^- = r_0^- \cup_x r_1^-$ then represent arcs in the projections $\pi_Y(\lambda^+)$ and $\pi_Y(\lambda^-)$ and after a slight perturbation these leaves have disjoint interiors. Hence, again we see that $d_Y(\lambda^+, \lambda^-) \leq 1$. This completes the proof. \square

5.2. Projections and τ -compatibility. We now show how to associate to a subsurface Y of large projection a representative of Y *within* the veering triangulation. This will later be used to prove that such a subsurface induces a pocket of the veering triangulation τ .

Call a subsurface $Y \subset X$ τ -*compatible* if the map $\hat{\iota}_q : Y \rightarrow \bar{X}_Y$ is homotopic to a map $\hat{\iota}_\tau : Y \rightarrow \bar{X}_Y$ such that

- (1) $\hat{\iota}_\tau$ is an embedding on $Y' = Y \setminus \partial_0 Y$,
- (2) $\hat{\iota}_\tau$ is τ -simplicial, and in particular $\hat{\iota}_\tau$ takes $\partial' Y$ to a multicurve in $\bar{X}_Y \setminus \tilde{\mathcal{P}}_Y$ composed of unions of τ -edges, and $\partial_0 Y$ to points of $\tilde{\mathcal{P}}_Y$.
- (3) The projection ι_τ to X is an embedding on the interior of Y .

We will show that when $d_Y(\lambda^-, \lambda^+)$ is sufficiently large, the subsurface Y is τ -compatible and in this case we set $\partial_\tau Y = \iota_\tau(\partial' Y) \subset \tau$. We call $\partial_\tau Y$ the τ -*boundary* of Y . Similar to the situation of a q -compatible subsurface, if Y is τ -compatible then a component of $X \setminus \partial_\tau Y$ is an open subsurface isotopic to the interior of Y ; this is the image $\iota_\tau(\text{int}(Y))$ and is denoted $\text{int}_\tau(Y)$.

Theorem 5.2 (τ -Compatibility). *Let $Y \subset X$ be an essential subsurface.*

- (1) *If Y is nonannular and $d_Y(\lambda^+, \lambda^-) > 0$, then Y is τ -compatible.*
- (2) *If Y is an annulus and $d_Y(\lambda^+, \lambda^-) > 1$, then Y is τ -compatible.*

Proof. By [Proposition 5.1](#), $\hat{\iota}_q : Y \rightarrow \bar{X}_Y$ is an embedding on Y' . Let Y_q denote its image. We first suppose that Y is not an annulus.

Give $\partial' Y$ the transverse orientation pointing into Y . For any saddle connection σ in $\hat{\iota}_q(\partial' Y)$ and any triangle $t \in \mathcal{T}(\sigma)$ pointing into Y , note that the singularities of \bar{X}_Y in ∂t are *not* completion points of \bar{X}_Y , that is they do not correspond to punctures of X . This is because any completion point lying in t is the endpoint of leaves l^\pm of λ^\pm whose initial segments lie in t . These leaves correspond to essential proper arcs of X_Y which are homotopic giving $d_Y(\lambda^-, \lambda^+) = 0$, a contradiction.

Hence, $\mathbf{t}(\hat{\iota}_q|_{\partial' Y})$ is homotopic to $\hat{\iota}_q|_{\partial' Y}$ in $\bar{X}_Y \setminus \tilde{\mathcal{P}}_Y$ by pushing across the polygonal regions given by [Lemma 4.2](#). This gives a homotopy of $\hat{\iota}_q$ to a map $\mathbf{t}\hat{\iota}_q$, which we claim is still an embedding. (Note that, in the case that X is fully-punctured, $\mathbf{t}\hat{\iota}_q = \hat{\iota}_q$, by the observation in the previous paragraph.)

To prove the claim, let C be a component of the preimage of $\hat{i}_q(Y')$ in the completion \hat{X} of the universal cover. If α is a geodesic segment in ∂C , the triangles used in the hull construction are attached to α on the C side. If such a triangle T intersects a triangle T' from a different segment α' , they overlap as in **Figure 16**. Their overlap contains a rectangle, and two arcs l^+, l^- of λ^+ and λ^- passing through that rectangle must intersect both α and α' . These arcs are at distance 0 in $\mathcal{A}(Y)$, since they can be isotoped to each other rel ∂Y . Hence $d_Y(\lambda^-, \lambda^+) = 0$, contradicting the hypothesis.

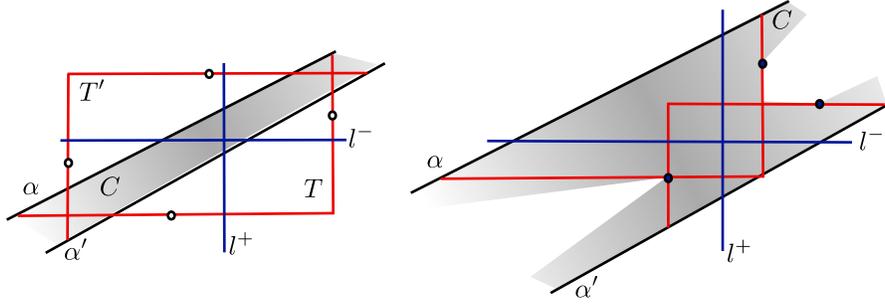


FIGURE 16. An overlap of two hull triangles. Note that the singularities defining the triangles can appear outside C (left figure) or inside C (right figure). Any completion point in a hull triangle does not correspond to a puncture in $\tilde{\mathcal{P}}_Y$.

We conclude that the polygonal regions of our homotopy are embedded and disjoint, and therefore that $\mathbf{t}\hat{i}_q$ remains an embedding.

Now orient $\partial'Y$ in the opposite direction, pointing out of the surface, and apply \mathbf{t} again, this time to $\mathbf{t}\hat{i}_q$. The triangles in the construction now extend outside the surface, and the result of the operation is the rectangle hull $\mathbf{r}(\mathbf{t}\hat{i}_q(\partial'Y))$, which is therefore composed of τ -edges. Composing with the homotopy to this new map of the boundary, we obtain our final map \hat{i}_τ .

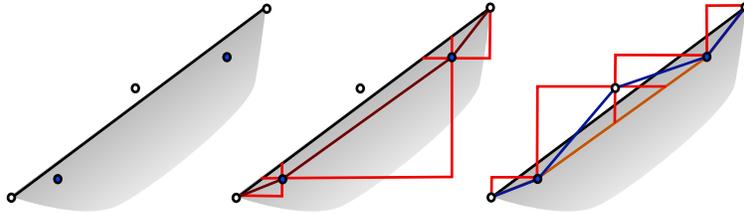


FIGURE 17. An inner \mathbf{t} followed by outer \mathbf{t} yields τ -edges.

Let C and C' be components of the lift of $\mathbf{t}\hat{i}_q(Y)$ to \hat{X} . The second \mathbf{t} construction appends triangles to the outside of C and C' , and again each edge α, α' is moved

through a polygon to its image. To see that these polygons are disjoint, consider two triangles T, T' that overlap. Each triangle is part of a singularity-free rectangle R, R' , and the singularities on the boundaries are forced to lie in the region outside C, C' between the hypotenuses of T and T' . See [Lemma 4.1](#). It follows that the regions between α, α' and their \mathbf{t} -images have disjoint interiors. This shows that ι_τ embeds the interior of Y in X . Moreover, in \bar{X}_Y , $\hat{\iota}_\tau$ is an embedding on all of Y' : if an edge b is in the \mathbf{t} image of both α and α' then it is transverse to a vertical geodesic arc whose endpoints lie on the boundary of the lift to \hat{X} of the q -hull $\hat{\iota}_q(Y)$. A failure of $\hat{\iota}_\tau$ to embed would correspond to such a vertical arc that, projected to \bar{X}_Y , bounds a bigon with an arc of $\hat{\iota}_q(\partial'Y)$. This contradicts uniqueness of geodesics in a homotopy class.

Now suppose that Y is an annulus. Then $\hat{\iota}_q(Y)$ is the (nondegenerate) maximal flat cylinder of \bar{X}_Y by [Proposition 5.1](#). We claim that $\mathbf{t}(\hat{\iota}_q(\partial Y)) = \hat{\iota}_q(\partial Y)$: Otherwise, if σ is a saddle connection on the boundary of the flat annulus $\hat{\iota}_q(Y)$, and t is a triangle pointing into the annulus with hypotenuse a proper subsegment of σ , then t must encounter a singularity or puncture q on the other side of the annulus. A variation on the argument in the annulus case of [Proposition 5.1](#) then produces vertical and horizontal leaves passing through q which have disjoint representatives, and hence $d_Y(\lambda^+, \lambda^-) \leq 1$.

The proof now proceeds just as in the nonannular case. \square

6. EMBEDDED POCKETS OF THE VEERING TRIANGULATION AND BOUNDED PROJECTIONS

In this section, let X be fully-punctured with respect to the foliations λ^\pm of a pseudo-Anosov $f : X \rightarrow X$, and let M be the mapping torus. Note that every fiber associated to the fibered face \mathcal{F} of X must also be fully-punctured because they are transverse to the same suspension flow. In this situation we say that M is a hyperbolic 3-manifold with a *fully-punctured fibered face*.

We now prove our two main theorems on the structure of subsurface projections in a fully-punctured fibered face, [Theorem 1.1](#) and [Theorem 1.2](#). The main tools in the proof are the structure and embedding theorems for pockets associated with high-distance subsurfaces, which we develop below.

6.1. Projections and τ -compatible subsurfaces. We begin by discussing projection to τ -compatible subsurfaces.

Lemma 6.1. *Let Y and Z be τ -compatible subsurfaces of X and let $K \subset X$ be a disjoint collection of edges from τ . Then*

- (1) *If K meets $\text{int}_\tau(Y)$, then $\pi_Y(K) \neq \emptyset$, and $\text{diam}_Y(\pi_Y(K)) \leq 1$.*
- (2) *If Y and Z are disjoint, then so are $\text{int}_\tau(Y)$ and $\text{int}_\tau(Z)$.*
- (3) *If Y and Z overlap, then $\text{diam}_Z(\partial Y, \partial_\tau Y) \leq 1$.*
- (4) *The subsurface $\text{int}_\tau(Y)$ is in minimal position with the foliations λ^\pm . In particular, the arcs of $\text{int}_\tau(Y) \cap \lambda^\pm$ agree with the arcs of $\pi_Y(\lambda^\pm)$.*

Proof. First, let e be a τ -edge which meets $\text{int}_\tau(Y)$. By lifting to \bar{X}_Y as in [Lemma 2.6](#) and using the local CAT(0) geometry, it is clear that so long as e meets $\text{int}_q(Y)$ then

it does so essentially. If e meets $\text{int}_\tau(Y)$ but not $\text{int}_q(Y)$, then e meets the interior of a disk D in X whose boundary is a union of saddle connections – one of these saddle connections is $\sigma \subset \partial_q Y$ and the others are $\mathbf{t}(\sigma) = \mathbf{r}(\sigma) \subset \partial_\tau Y$ (recall from the proof of [Theorem 5.2](#) that, since X is fully-punctured, the inner \mathbf{t} step in the construction of $\hat{\iota}_\tau$ is the identity, and the outer \mathbf{t} step therefore yields $\mathbf{r}(\sigma)$). Let R be the singularity-free rectangle spanned by e . If e is contained in D then R can be extended to a rectangle whose diagonal lies in σ , and hence e is one of the edges of $\mathbf{r}(\sigma)$; but this contradicts the assumption that e meets $\text{int}_\tau(Y)$. Thus e crosses some edge f of $\mathbf{r}(\sigma)$. However, f is contained in a singularity-free triangle whose hypotenuse lies along σ and so σ must cross the rectangle R either top-to-bottom or side-to-side. In either case, we see that e crosses $\sigma \subset \partial_q(Y)$, and therefore has a well-defined projection to $\mathcal{A}(Y)$. Item (1) is then immediate since K is a disjoint collection of essential arcs of $\mathcal{A}(X)$.

For item (2), first note that when Y and Z are disjoint subsurfaces of X , the interiors $\text{int}_q(Y)$ and $\text{int}_q(Z)$ are also disjoint. This follows from [Corollary 2.4](#) and the q -hulls construction in [Lemma 2.6](#). More precisely, let Λ_Y and Λ_Z be the limit sets of Y and Z in $\partial\mathbb{H}^2$ (using our identifications from [Section 2.3](#)). Since Y and Z do not intersect, Λ_Y and Λ_Z do not link in $\partial\mathbb{H}^2$ and so $CH_q(\Lambda_Y)$ and $CH_q(\Lambda_Z)$ have disjoint interiors by [Corollary 2.4](#). This implies that $\text{int}_q(Y)$ and $\text{int}_q(Z)$ are disjoint in X .

To obtain $\partial_\tau Y$ from $\partial_q Y$ and $\partial_\tau Z$ from $\partial_q Z$, we apply the rectangle hull $\mathbf{r}(\cdot)$ as in [Theorem 5.2](#). By [Lemma 4.1](#), $\partial_\tau Y$ and $\partial_\tau Z$ are noncrossing and since the transitions from $\text{int}_q(Y)$ to $\text{int}_\tau(Y)$ and from $\text{int}_\tau(Y)$ to $\text{int}_\tau(Z)$ are via isotopy in X through singularity-free disks, the interiors $\text{int}_\tau(Y)$ and $\text{int}_\tau(Z)$ are disjoint.

Since $\text{int}_\tau(Y)$ is isotopic to the interior of Y , ∂Y has a representative disjoint from the collection of saddle connections in $\partial_\tau Y$. Hence, $\text{diam}_Z(\partial Y, \partial_\tau Y) \leq 1$ proving item (3). For item (4), note first that it is immediate if we replace $\text{int}_\tau(Y)$ with $\text{int}_q(Y)$ since we may again use the local CAT(0) geometry in \bar{X}_Y and the fact that λ^\pm are geodesic. The statement for $\text{int}_\tau(Y)$ then follows from the fact that the homotopy from $\partial_q Y$ to $\partial_\tau Y$ can be taken to move either along vertical or along horizontal leaves. \square

6.2. Pockets for a τ -compatible subsurface. Suppose that $Y \subset X$ is τ -compatible. By [Corollary 3.6](#), the set $T(\partial_\tau Y)$ of sections containing $\partial_\tau Y$ contains a top and a bottom section, denoted $T^+ = T^+(\partial_\tau Y)$ and $T^- = T^-(\partial_\tau Y)$, which between them bound a number of pockets. Our assumption on $d_Y(\lambda^-, \lambda^+)$ will imply that one of these pockets is isotopic to a thickening of Y , as explained in the following proposition:

Proposition 6.2 (Pockets in τ). *Let (X, q) be fully-punctured and $Y \subset X$ an essential nonannular subsurface.*

- (1) *If $d_Y(\lambda^-, \lambda^+) > 0$ then $d_Y(T^+, \lambda^+) = d_Y(T^-, \lambda^-) = 0$.*
- (2) *If $d_Y(\lambda^-, \lambda^+) > 2$ then T^+ and T^- bound a pocket U_Y whose interior is isotopic to a thickening of $\text{int}(Y)$.*

When Y is an annulus,

- (1) If $d_Y(\lambda^-, \lambda^+) > 1$ then $d_Y(T^+, \lambda^+) = d_Y(T^-, \lambda^-) = 1$.
 (2) If $d_Y(\lambda^-, \lambda^+) > 4$ then T^+ and T^- bound a pocket U_Y whose interior is isotopic to a thickening of $\text{int}(Y)$.

Proof. Begin with the following lemma:

Lemma 6.3. *Suppose that $Y \subset X$ is τ -compatible, let e be an edge of $\partial_\tau Y$ and let f be a τ -edge crossing e with $f > e$. Then $d_Y(f, \lambda^+) \leq 1$ if Y is an annulus and $d_Y(f, \lambda^+) = 0$ otherwise. Similarly if $f < e$ then the same statement holds for $d_Y(f, \lambda^-)$.*

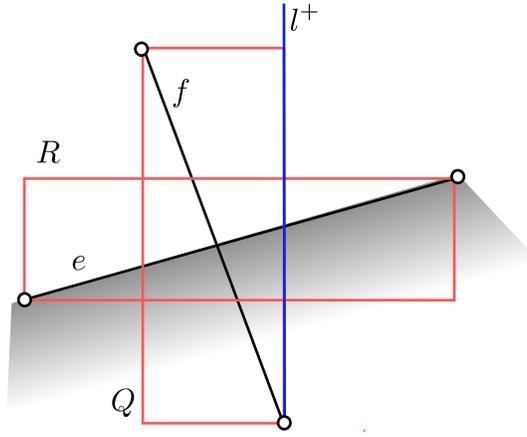


FIGURE 18. When $f > e$, the edge l^+ of Q represents $\pi_Y(\lambda^+)$ and is disjoint from f .

Proof. Let R be the rectangle spanned by e and Q the rectangle spanned by f . Since $f > e$, Q must cross R from top to bottom (see Section 3). The edge e cuts Q into two sides, at least one of which initially meets the image of Y . Let s be the corner of Q on this side which is an endpoint of f , and let l be the leaf of λ^+ emanating from s and running vertically along ∂Q (Figure 18). This leaf continues to e without intersecting f , and since X is fully punctured it gives us a representative of $\pi_Y(\lambda^+)$ (Lemma 6.1). Hence $d_Y(f, \lambda^+) = 0$ if Y is nonannular.

If Y is an annulus, we lift the picture to the annular cover, where we note that the leaf l , continued to infinity, cannot intersect f without meeting Q , and hence e , again. Since l can only meet $\partial_q Y$ once in the annular cover, we conclude it is disjoint from f and so $d_Y(f, \lambda^+) = 1$.

The case $f < e$ is similar. □

Let Y be nonannular. Note that by definition the only upward-flippable edges in T^+ must lie in $\partial_\tau Y$. Let e be such an edge and consider the single flip move that replaces e with an edge f . Then $f > e$, so by Lemma 6.3, $d_Y(f, \lambda^+) = 0$. On the

other hand f and e are diagonals of a quadrilateral E made of edges of T^+ , at least one of which, e' , gives the same element of $\mathcal{A}(Y)$ as f . Hence $d_Y(T^+, \lambda^+) = 0$.

If Y is an annulus, we note that e' and the vertical leaf in the proof of [Lemma 6.3](#) give adjacent vertices of $\mathcal{A}(Y)$, so $d_Y(T^+, \lambda^+) \leq 1$.

To prove the statements about pockets, let $K = T^+ \cap T^-$, viewed as a subcomplex of X . If $\text{int}_\tau(Y)$ contains an edge of K then, together with what we have proved, we obtain $d_Y(\lambda^+, \lambda^-) \leq 2$ when Y is nonannular, and $d_Y(\lambda^+, \lambda^-) \leq 4$ when Y is an annulus. By our hypotheses this does not happen, so we conclude that $\text{int}_\tau(Y)$ is the base of a pocket U_Y . \square

6.3. Isolated pockets and projection bounds. Let X be a fiber in \mathcal{F} , and let Y be a τ -compatible subsurface of X such that $d_Y(\lambda^-, \lambda^+) > 3$. An *isolated pocket* for Y in $(X \times \mathbb{R}, \tau)$ is a subpocket $V = V_Y$ of U_Y with base $\text{int}_\tau(Y)$ such that

- (1) For each edge e of V which is not contained in $\partial_\tau Y$,

$$d_Y(e, \lambda^\pm) \geq 3$$

if Y is nonannular, and

$$d_Y(e, \lambda^\pm) \geq 4$$

if Y is an annulus.

- (2) Denoting V^\pm the top and bottom of V with their induced triangulations,

$$d_Y(V^-, V^+) \geq 1.$$

Note that condition (2) guarantees that $\text{int}(V_Y) \cong \text{int}_\tau(Y) \times (0, 1)$ just as in [Proposition 6.2](#). The next lemma shows that for Y with $d_Y(\lambda^-, \lambda^+)$ sufficiently large, Y has an isolated pocket with $d_Y(V^-, V^+)$ roughly $d_Y(\lambda^-, \lambda^+)$.

Lemma 6.4. *Suppose that Y is a nonannular subsurface of X with $d_Y(\lambda^-, \lambda^+) > 8$. Then Y has an isolated pocket V with $d_Y(V^-, V^+) \geq d_Y(\lambda^-, \lambda^+) - 8$.*

If Y is an annulus with $d_Y(\lambda^-, \lambda^+) > 10$. Then Y has an isolated pocket V with $d_Y(V^-, V^+) \geq d_Y(\lambda^-, \lambda^+) - 10$.

Proof. Let $c = 4$ if Y is an annulus and $c = 3$ otherwise, so that we have $d_Y(\lambda^+, \lambda^-) > 2c + 2$. Since the pocket $U = U_Y$ is connected, there is a sequence of sections $T^- = T_0, T_1, \dots, T_N = T^+$ in $T(\partial_\tau Y)$ such that T_{i+1} differs from T_i by an upward diagonal exchange. From [Proposition 6.2](#), we know that $d_Y(T^-, \lambda^-) \leq 1$ and $d_Y(T^+, \lambda^+) \leq 1$. Let $0 < a < N$ be largest integer such that $d_Y(T_{a-1}, \lambda^-) < c$; hence $d_Y(T_i, \lambda^-) \geq c$ for all $i \geq a$. Now let $b < N$ be the smallest integer greater than a such that $d_Y(T_{b+1}, \lambda^+) < c$; then $d_Y(T_i, \lambda^+) \geq c$ for all $a \leq i \leq b$.

Note that these indices exist since $d_Y(\lambda^-, \lambda^+) \geq 2c + 1$.

Now let V be the pocket between T_a and T_b with base $\text{int}_\tau(Y)$ and note that V is a subpocket of U . Any edge e of V not contained in $\partial_\tau Y$ is contained in a section $T_i \in T(\partial_\tau Y)$ for $a \leq i \leq b$. Since we have $d_Y(T_i, \lambda^\pm) \geq c$, we have $d_Y(e, \lambda^\pm) \geq c$. Thus it only remains to get a lower bound on $d_Y(V^+, V^-)$.

The triangle inequality (and diameter bound on T_a and T_b) gives

$$d_Y(V^-, V^+) = d_Y(T_a, T_b) \geq d_Y(\lambda^-, \lambda^+) - 2c - 2 \geq 1,$$

which completes the proof. \square

The following proposition shows that isolated pockets coming from either disjoint or overlapping subsurfaces of X have interiors which do not meet.

Proposition 6.5 (Disjoint pockets). *Suppose that Y and Z are subsurfaces of X with isolated pockets V_Y and V_Z . Then, up to switching Y and Z , either Y is nested in Z , or the isolated pockets V_Y and V_Z have disjoint interiors in $X \times \mathbb{R}$.*

Proof. If the subsurface Y and Z are disjoint, then $\text{int}_\tau(Y)$ and $\text{int}_\tau(Z)$ are also disjoint by [Lemma 6.1](#). Hence, the maximal pockets U_Y and U_Z have disjoint interiors by definition.

Now suppose that Y is not an annulus. We claim that if Y and Z overlap then either

$$d_Y(\partial_\tau Z, \lambda^+) \leq 1 \text{ or } d_Y(\partial_\tau Z, \lambda^-) \leq 1.$$

To see this, first note that there is some edge f contained in $\text{int}_\tau(Z)$ such that f crosses some edges of $\partial_\tau Y$. Otherwise, every triangulation of $\text{int}_\tau(Z)$ by τ -edges contains edges from $\partial_\tau Y$. Hence, by [Proposition 6.2](#)

$$d_Z(\lambda^-, \lambda^+) \leq 2 + \text{diam}_Z(\partial_\tau Y) \leq 3,$$

contradicting our assumption on the subsurface Z . Now if f intersects an edge e of $\partial_\tau Y$ and $f > e$, then by [Lemma 6.3](#), $d_Y(\partial_\tau Z, \lambda^+) \leq d_Y(\partial_\tau Z, f) \leq 1$. If $f < e$ then [Lemma 6.3](#) gives $d_Y(\partial_\tau Z, \lambda^-) \leq d_Y(\partial_\tau Z, f) \leq 1$.

Finally, suppose that e is an edge of $U_Y \cap U_Z$ which is not contained in $\partial_\tau Y \cup \partial_\tau Z$. Then e , as a τ -edge in X , is disjoint from $\partial_\tau Z$ and so $d_Y(e, \lambda^+) \leq 2$ or $d_Y(e, \lambda^-) \leq 2$. Hence e cannot be contained in V_Y . We conclude that $V_Y \cap V_Z \subset \partial_\tau Y \cup \partial_\tau Z$. This completes the proof when Y is not an annulus.

When Y is an annulus, then a similar argument using the annular case of [Lemma 6.3](#) shows that if Y and Z overlap then either

$$d_Y(\partial_\tau Z, \lambda^+) \leq 2 \text{ or } d_Y(\partial_\tau Z, \lambda^-) \leq 2.$$

Hence, if e is an edge of $U_Y \cap U_Z$ which is not contained in $\partial_\tau Y \cup \partial_\tau Z$, then $d_Y(e, \lambda^\pm) \leq 3$. So again e cannot be contained in V_Y and we conclude that $V_Y \cap V_Z \subset \partial_\tau Y \cup \partial_\tau Z$ as required. \square

We next prove that isolated pockets embed into the fibered manifold M . This is [Theorem 1.3](#), which we restate here in more precise language.

Theorem 1.3 (Embedding the pocket). *Suppose Y is a subsurface of a fully-punctured fiber X with $d_Y(\lambda^-, \lambda^+) > \beta$, where $\beta = 8$ if Y is nonannular and $\beta = 10$ if Y is an annulus. Then Y has an isolated pocket V_Y in $X \times \mathbb{R}$, and the covering map $X \times \mathbb{R} \rightarrow M$ restricts to an embedding of the subcomplex $V_Y \rightarrow M$.*

Proof. Note that if T is a section of τ , then $\Phi(T)$ is the section of τ whose corresponding triangulation of X is $f(T)$. Hence, $\Phi(T(\partial_\tau Y)) = T(\partial_\tau f(Y))$.

By [Lemma 6.4](#), Y has an isolated pocket $V = V_Y$. Note that V embeds into M if and only if it is disjoint from its translates $V_i = \Phi^i(V)$ for each $i \neq 0$. By the remark above, each V_i is itself an isolated pocket for the subsurface $Y_i = f^i(Y)$, and any two

of these subsurfaces are either disjoint or overlap in X . Hence, by [Proposition 6.5](#) the isolated pockets V_i are disjoint as required. \square

We will now prove [Theorem 1.1](#), whose statement we recall here:

Theorem 1.1 *Let M be a hyperbolic 3-manifold with fully-punctured fibered face \mathcal{F} and veering triangulation τ . For any subsurface W of any fiber of \mathcal{F} ,*

$$\alpha \cdot (d_W(\lambda^-, \lambda^+) - \beta) < |\tau|,$$

where $|\tau|$ is the number of tetrahedra in τ , $\alpha = 1$ and $\beta = 10$ when W is an annulus and $\alpha = 3|\chi(W)|$ and $\beta = 8$ when W is not an annulus.

Proof. Suppose that W is any nonannular subsurface of any fiber F in $\mathbb{R}_+\mathcal{F}$. We may assume that $d_W(\lambda^-, \lambda^+) > 8$. Then [Lemma 6.4](#) implies that W has an isolated pocket V_W in $(F \times \mathbb{R}, \tau)$ such that $d_W(V_W^-, V_W^+) \geq d_Y(\lambda^-, \lambda^+) - 8$. By [Theorem 1.3](#), the isolated pocket $V_W \subset (F \times \mathbb{R}, \tau)$ embeds into (M, τ) . Hence $|V_W| \leq |\tau|$, where $|V_W|$ denotes the number of tetrahedra of V_W . Now each tetrahedron of V_W corresponds to a diagonal exchange between the triangulations V_W^- and V_W^+ of W_τ and each diagonal exchange replaces a single edge of the triangulation. There are at least $3|\chi(W)| + 1$ non-boundary edges to each triangulation of W , and the diameter in $\mathcal{A}(W)$ of an ideal triangulation is 1, so we conclude

$$\begin{aligned} (1) \quad |\tau| &\geq |V_W| = \#\{\text{diagonal exchanges from } V_W^- \text{ to } V_W^+\} \\ &> 3|\chi(W)| \cdot d_W(V^-, V^+) \\ &\geq 3|\chi(W)| \cdot (d_W(\lambda^-, \lambda^+) - 8). \end{aligned}$$

This completes the proof when W is nonannular.

When W is an annulus, we use the annular case of [Lemma 6.4](#) to obtain an isolated pocket V_W in $(F \times \mathbb{R}, \tau)$ such that $d_W(V_W^-, V_W^+) \geq d_Y(\lambda^-, \lambda^+) - 10$. Noting that a triangulation of the annulus contains at least 2 (non-boundary) edges, the same argument implies that

$$\begin{aligned} |\tau| &\geq |V_W| = \#\{\text{diagonal exchanges from } V_W^- \text{ to } V_W^+\} \\ &> d_W(V^-, V^+) \\ &\geq d_W(\lambda^-, \lambda^+) - 10, \end{aligned}$$

as required. \square

6.4. Sweeping through embedded pockets. We are now ready to prove [Theorem 1.2](#), whose statement we reproduce below. This theorem relates subsurfaces of large projections among different fibers of a fixed face.

Theorem 1.2 *Let M be a hyperbolic 3-manifold with fully-punctured fibered face \mathcal{F} and suppose that S and F are each fibers in $\mathbb{R}_+\mathcal{F}$. If W is a subsurface of F , then either W is isotopic along the flow to a subsurface of S , or*

$$3|\chi(S)| \geq d_W(\lambda^-, \lambda^+) - \beta,$$

where $\beta = 10$ if W is an annulus and $\beta = 8$ otherwise.

Recall from [Lemma 2.8](#) that we can identify $d_W(\lambda^+, \lambda^-)$ with $d_W(\Lambda^+, \Lambda^-)$, agreeing with the statement given in the introduction.

We will require the following lemma, which essentially states that immersed subsurfaces with large projection are necessarily covers of subsurfaces. Recall that in [Section 2.2](#) we defined the distance $d_W(\lambda^+, \lambda^-)$ when W is the compact core of a cover $X_\Gamma \rightarrow X$ corresponding to a finitely generated subgroup $\Gamma \leq \pi_1(X)$.

Lemma 6.6 (Immersion to cover). *Suppose that X is a fully-punctured surface. Let Γ be a finitely generated subgroup of $\pi_1(X)$ and let W be the compact core of the cover $X_\Gamma \rightarrow X$. If W is nonannular and $d_W(\lambda^-, \lambda^+) > 4$ or if W is an annulus and $d_W(\lambda^-, \lambda^+) > 6$, then there is a subsurface Y of X such that $W \rightarrow X$ is homotopic to a finite cover $W \rightarrow Y \subset X$.*

In particular, Γ is a finite index subgroup of $\pi_1(Y)$.

Proof. Let $p: \check{X} \rightarrow X$ be a finite cover to which $W \rightarrow X$ lifts to an embedding $W \rightarrow \check{X}$ and identify W with its image in \check{X} . Lift q along with the veering triangulation to $(\check{X} \times \mathbb{R}, \tau)$. By our assumption on distance, [Theorem 5.2](#) and [Proposition 3.3](#), $T_{\check{X}}(\partial_\tau W)$ is nonempty and connected.

First suppose that W is not an annulus. If \check{T} is a section of $(\check{X} \times \mathbb{R}, \tau)$ with an edge f such that $f > e$ for an edge e of $\partial_\tau W$, then [Lemma 6.3](#) implies that $d_W(T, \lambda^+) = 0$. Similarly if $f < e$ then $d_W(T, \lambda^-) = 0$. Hence, if T is any section of $(X \times \mathbb{R}, \tau)$ such that $d_W(T, \lambda^\pm) \geq 1$, then its lift $\check{T} = p^{-1}(T)$ to \check{X} must contain the edges of $\partial_\tau W$ and so $\check{T} \in T_{\check{X}}(\partial_\tau W)$.

From this, we conclude that the image of $\partial_\tau W$ in X does not have self crossings, and that for each section T of $(X \times \mathbb{R}, \tau)$ with $d_W(T, \lambda^\pm) \geq 1$,

$$p^{-1}(T) \in T_{\check{X}}(p^{-1}(p(\partial_\tau W))).$$

We claim now that there can be no edge e in $p^{-1}(p(\partial_\tau W))$ which is contained in $\text{int}_\tau(W)$. Such an edge would have a well-defined projection to $\mathcal{A}(W)$, and $d_W(p^{-1}(T), e) = 0$ whenever $d_W(T, \lambda^\pm) \geq 1$. But since we can sweep through $X \times \mathbb{R}$ with sections going from near λ^- to near λ^+ , this implies that $d_W(\lambda^\pm, e) \leq 2$, which contradicts our hypothesis that $d_W(\lambda^+, \lambda^-) > 4$. We conclude that $p^{-1}(p(\partial_\tau W)) \cap W_\tau = \partial_\tau W$ and therefore that $\text{int}_\tau(W) \rightarrow X$ covers a subsurface Y of X , as required.

When W is an annulus, one proceeds exactly as above using the annular version of [Lemma 6.3](#). \square

Proof of [Theorem 1.2](#). We may assume that $d_W(\lambda^-, \lambda^+) > \beta$.

First suppose that $\pi_1(W)$ is contained in $\pi_1(S)$. Then by [Lemma 6.6](#), there is a subsurface Y of S such that $\pi_1(W) \leq \pi_1(Y)$ is a finite index subgroup; assume that the index of $\pi_1(W)$ in $\pi_1(Y)$ is $n \geq 1$. If $\eta_F: \pi_1(M) \rightarrow \mathbb{Z}$ denotes the homomorphism representing the cohomology class dual to F , then $\eta_F|_{\pi_1(Y)}$ vanishes on the index n subgroup $\pi_1(W)$. Since \mathbb{Z} is torsion-free we must have $n = 1$. Hence, $\pi_1(W) = \pi_1(Y)$. That W is isotopic along the flow in M to $Y \subset S$ can be seen by lifting W and Y to the cover $S \times \mathbb{R} \rightarrow M$.

Hence, we may suppose by [Lemma 2.9](#) that the image of any $S \rightarrow M$ homotopic to the fiber S intersects any isotope of $W \subset F$ essentially. Since $d_W(\lambda^-, \lambda^+) > \beta$, W has a nonempty isolated pocket $V_W \subset F \times \mathbb{R}$ which simplicially embeds into (M, τ) by [Theorem 1.3](#). Let $\{W_i\}$ denote a sequence of sections of V_W from V_W^- to V_W^+ with W_{i+1} differing from W_i by an upward diagonal flip. Also, fix a simplicial map $f: S \rightarrow (M, \tau)$ which is obtained by composing a section of $(S \times \mathbb{R}, \tau)$ with the covering map $S \times \mathbb{R} \rightarrow M$.

Note that for each i , $f(S)$ meets at least one edge of the interior of W_i . Otherwise, the image of S in M misses the interior of W_i contradicting our assumption. In fact, even more is true: Call a component c of $f(S) \cap W_i$ *removable* if the triangles of $f(S)$ incident to the edges of c lie locally to one side of W_i in M . If c is removable, then there is an isotopy of W_i supported in a neighborhood of c which removes c from the intersection $f(S) \cap W_i$. Hence, if we denote E_i the edges of $f(S) \cap W_i$ which do not lie in removable components, then E_i must be nonempty for each i .

We claim that for each i , E_i shares an edge with E_{i+1} . Otherwise, the tetrahedron corresponding to the diagonal exchange from W_i to W_{i+1} has E_i as its bottom edge and E_{i+1} as its top edge. But then both of these edges must be removable since pushing the bottom two faces of the tetrahedron slightly upward makes that intersection disappear, and similarly for the top. This contradicts our above observation and establishes that E_i and E_{i+1} have a common edge.

We obtain a sequence in $\mathcal{A}(W)$,

$$V_W^- \supset E_0, E_1, \dots, E_n \subset V_W^+,$$

having the property that for each edge e_i of E_i there is an edge e_{i+1} of E_{i+1} such that e_i and e_{i+1} are disjoint. We conclude that the number of distinct edges in the sequence E_0, E_1, \dots, E_n is at least $d_W(V_W^-, V_W^+)$. Combining this with the fact that the number of edges in an ideal triangulation of S is $3|\chi(S)|$ and [Lemma 6.4](#), we see that

$$3|\chi(S)| \geq d_W(V_W^-, V_W^+) \geq d_W(\lambda^-, \lambda^+) - \beta,$$

as required. □

We conclude the paper by recording the following corollary of [Lemma 6.6](#) and the proof of [Theorem 1.2](#).

Corollary 6.7. *Let M be a hyperbolic manifold with fully-punctured fibered face \mathcal{F} . Let W be a subsurface of a fiber $F \in \mathbb{R}_+\mathcal{F}$ such that $d_W(\Lambda^+, \Lambda^-) > 4$ if W is nonannular and $d_W(\Lambda^+, \Lambda^-) > 6$ if W is an annulus. If S is any fiber in $\mathbb{R}_+\mathcal{F}$ such that $\pi_1(W) < \pi_1(S)$, then W is isotopic to a subsurface of S .*

REFERENCES

- [Ago11] Ian Agol, *Ideal triangulations of pseudo-Anosov mapping tori*, *Topology and geometry in dimension three* **560** (2011), 1–17.
- [Ago12] ———, *Comparing layered triangulations of 3-manifolds which fiber over the circle*, *MathOverflow discussion*, <http://mathoverflow.net/questions/106426>, 2012.
- [BCM12] J. Brock, R. Canary, and Y. Minsky, *The classification of Kleinian surface groups, II: The ending lamination conjecture*, *Ann. of Math.* **176** (2012), no. 1, 1–149. MR 2925381

- [BKMM12] Jason Behrstock, Bruce Kleiner, Yair Minsky, and Lee Mosher, *Geometry and rigidity of mapping class groups*, *Geom. Topol.* **16** (2012), 781–888.
- [BS05] David Bachman and Saul Schleimer, *Surface bundles versus Heegaard splittings*, *Communications in Analysis and Geometry* (2005), no. 5, 903 – 928.
- [CC00] Alberto Candel and Lawrence Conlon, *Foliations II*, American Mathematical Society Providence, 2000.
- [FLM11] B. Farb, C. J. Leininger, and D. Margalit, *Small dilatation pseudo-Anosov homeomorphisms and 3-manifolds*, *Adv. Math.* **228** (2011), no. 3, 1466–1502. MR 2824561
- [Fri82] David Fried, *The geometry of cross sections to flows*, *Topology* **21** (1982), no. 4, 353–371.
- [Gué15] François Guéritaud, *Veering triangulations and the Cannon-Thurston map*, *ArXiv:1506.03387*, 2015.
- [Har02] K. Hartshorn, *Heegaard splittings of Haken manifolds have bounded distance*, *Pacific J. Math.* **204** (2002), no. 1, 61–75. MR MR1905192 (2003a:57037)
- [JMM10] J. Johnson, Y. Minsky, and Y. Moriah, *Heegaard splittings with large subsurface distances*, *Algebr. Geom. Topol.* **10** (2010), no. 4, 2251–2275. MR 2745671 (2012g:57038)
- [McM00] Curtis T McMullen, *Polynomial invariants for fibered 3-manifolds and Teichmüller geodesics for foliations*, *Annales scientifiques de l’Ecole normale supérieure* **33** (2000), no. 4, 519–560.
- [Min10] Yair Minsky, *The classification of Kleinian surface groups, I: Models and bounds*, *Ann. of Math.* (2010), 1–107.
- [MM00] Howard A. Masur and Yair N. Minsky, *Geometry of the complex of curves. II. Hierarchical structure*, *Geom. Funct. Anal.* **10** (2000), no. 4, 902–974.
- [MS13] Howard Masur and Saul Schleimer, *The geometry of the disk complex*, *Journal of the American Mathematical Society* **26** (2013), no. 1, 1–62.
- [Raf05] Kasra Rafi, *A characterization of short curves of a Teichmüller geodesic*, *Geometry & Topology* **9** (2005), no. 1, 179–202.
- [ST06] Martin Scharlemann and Maggy Tomova, *Alternate Heegaard genus bounds distance*, *Geom. Topol.* **10** (2006), 593–617 (electronic). MR 2224466
- [Thu86] William P Thurston, *A norm for the homology of 3-manifolds*, *Mem. Amer. Math. Soc.* **59** (1986), no. 339, 99–130.
- [TW15] Robert Tang and Richard Webb, *Shadows of Teichmüller disks in the curve graph*, *arXiv:1510.04259*, 2015.

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