CANNON–THURSTON MAPS FOR HYPERBOLIC FREE GROUP EXTENSIONS

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ABSTRACT. This paper gives a detailed analysis of the Cannon–Thurston maps associated to a general class of hyperbolic free group extensions. Let $\mathbb{F}$ denote a free groups of finite rank at least 3 and consider a *convex cocompact* subgroup $\Gamma \le \mathrm{Out}(\mathbb{F})$, i.e. one for which the orbit map from Γ into the free factor complex of $\mathbb{F}$ is a quasi-isometric embedding. The subgroup Γ determines an extension E_Γ of $\mathbb{F}$, and the main theorem of Dowdall–Taylor [DT1] states that in this situation E_Γ is hyperbolic if and only if Γ is purely atoroidal.

Here, we give an explicit geometric description of the Cannon–Thurston maps $\partial\mathbb{F} \to \partial E_\Gamma$ for these hyperbolic free group extensions, the existence of which follows from a general result of Mitra. In particular, we obtain a uniform bound on the multiplicity of the Cannon–Thurston map, showing that this map has multiplicity at most $2\,\mathrm{rank}(\mathbb{F})$. This theorem generalizes the main result of Kapovich and Lustig [KL5] which treats the special case where Γ is infinite cyclic. We also answer a question of Mahan Mitra by producing an explicit example of a hyperbolic free group extension for which the natural map from the boundary of Γ to the space of laminations of the free group (with the Chabauty topology) is not continuous.

1. Introduction

A remarkable paper of Cannon and Thurston [CT] proved that if M is a closed hyperbolic 3–manifold which fibers over the circle $\mathbb{S}^1$ with fiber S, then the inclusion of $\widetilde{S} = \mathbb{H}^2$ into $\widetilde{M} = \mathbb{H}^3$ extends to a continuous $\pi_1(S)$–equivariant surjective map from $\partial\mathbb{H}^2 = \mathbb{S}^1$ to $\partial\mathbb{H}^3 = \mathbb{S}^2$. In particular, the inclusion $\pi_1(S) \le \pi_1(M)$ of word-hyperbolic groups extends to a continuous map $\partial\pi_1(S) \to \partial\pi_1(M)$ of their Gromov boundaries. Though not published till 2007, this paper [CT] has been highly influential since its circulation as a preprint in 1984. Consequently, if the inclusion $\iota\colon H \to G$ of a word-hyperbolic subgroup H of a word-hyperbolic group G extends to a continuous (and necessarily H–equivariant and unique) map $\partial\iota\colon \partial H \to \partial G$, the map $\partial\iota$ came to be called the *Cannon–Thurston map*. It is easy to see that the Cannon–Thurston map always exists and is injective in the case that H is a quasiconvex subgroup of G; the above result of Cannon and Thurston [CT] provided the first nontrivial example of existence of $\partial\iota$ in the non-quasiconvex case.

Later the work of Mitra [Mit2, Mit3, Mit4] showed that the Cannon–Thurston map exists in several general situations corresponding to non-quasiconvex subgroups. In particular, Mitra proved [Mit2] that whenever

$$1 \longrightarrow H \longrightarrow G \longrightarrow \Gamma \longrightarrow 1 \tag{1}$$

is a short exact sequence of three infinite word-hyperbolic groups, then the Cannon–Thurston map $\partial\iota\colon \partial H \to \partial G$ exists and is surjective. Only recently did the work of Baker and Riley [BR1] produce the first example of a word-hyperbolic subgroup H of a word-hyperbolic group G for which the inclusion $H \le G$ does not extend to a Cannon–Thurston map. Analogs and generalizations of the Cannon–Thurston map have been studied in many other contexts, see for example [Kla, McM, Miy, LLR, LMS, Ger, Bow1, Bow2, MP, Mj2, Mj1, JKLO]. The best understood case concerns discrete isometric actions of surface groups on $\mathbb{H}^3$, where the most general results about Cannon–Thurston maps are due to Mj [Mj1].

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The main results from the theory of JSJ decomposition for word-hyperbolic groups (see [RS] for the original statement and [Lev] for a clarified version) imply that if we have a short exact sequence (1) of three infinite word-hyperbolic groups with H being torsion-free, then H is isomorphic to a free product of surface groups and free groups. Thus understanding the structure of the Cannon–Thurston map for such short exact sequences requires first studying in detail the cases where H is a surface group or a free group. The case of word-hyperbolic extensions of closed surface groups is closely related to the theory of convex cocompact subgroups of mapping class groups, and the structural properties of the Cannon–Thurston map in this setting are by now well understood; [LMS] for details.

In this paper we consider the case of hyperbolic extensions of free groups and specifically the class of hyperbolic extensions introduced by Dowdall and Taylor [DT1]. To describe this class, we henceforth fix a free group $\mathbb{F}$ of finite rank at least 3. Following Hamenstädt and Hensel [HH], we say that a finitely generated subgroup $\Gamma \leq \mathrm{Out}(\mathbb{F})$ is *convex cocompact* if the orbit map $\Gamma \to \mathcal{F}$ into the free factor graph $\mathcal{F}$ of $\mathbb{F}$ is a quasi-isometric embedding. Hyperbolicity of $\mathcal{F}$ [BF2] and the definition of $\mathcal{F}$ respectively imply convex cocompact subgroups of $\mathrm{Out}(\mathbb{F})$ are word-hyperbolic and that their infinite-order elements are all fully irreducible. We say that a subgroup $\Gamma \leq \mathrm{Out}(\mathbb{F})$ is *purely atoroidal* if every infinite-order element $\phi \in \Gamma$ is atoroidal (that is, no positive power of ϕ fixes a nontrivial conjugacy class in $\mathbb{F}$). Given any subgroup $\Gamma \leq \mathrm{Out}(\mathbb{F})$, the full pre-image E_Γ of Γ under the quotient map $\mathrm{Aut}(\mathbb{F}) \to \mathrm{Out}(\mathbb{F})$ fits into a short exact sequence

$$1 \longrightarrow \mathbb{F} \longrightarrow E_\Gamma \longrightarrow \Gamma \longrightarrow 1 \tag{2}$$

with kernel $\mathbb{F} \cong \mathrm{Inn}(\mathbb{F})$. The main result of [DT1] proves that E_Γ is hyperbolic whenever $\Gamma \leq \mathrm{Out}(\mathbb{F})$ is convex cocompact and purely atoroidal. From this, we see that if $\Gamma \leq \mathrm{Out}(\mathbb{F})$ is convex cocompact, then E_Γ is hyperbolic if and only if Γ is purely atoroidal. Moreover, there is a precise sense [TT] in which random finitely generated subgroups of $\mathrm{Out}(\mathbb{F})$ satisfy these hypotheses and so define hyperbolic extensions as in (2).

Our goal in this paper is to understand the Cannon–Thurston maps for hyperbolic extensions E_Γ of $\mathbb{F}$ corresponding to convex cocompact subgroups Γ of $\mathrm{Out}(\mathbb{F})$. Such extensions vastly generalize the hyperbolic free-by-cyclic groups with fully irreducible monodromy whose Cannon–Thurston maps were explored in detail by Kapovich and Lustig in [KL5]. Our analysis extends many results of [KL5] and gives an explicit description of the Cannon–Thurston map in the general setting of purely atoroidal convex cocompact Γ. Moreover, this description reveals quantitative global and local features of the map and allows us to address multiple conjectures regarding Cannon–Thurston maps in this setting. For example, we answer a question of Swarup (which appears as Question 1.20 on Bestvina's list) by establishing uniform finiteness of the fibers of $\partial\iota\colon \partial\mathbb{F} \to \partial E_\Gamma$.

Theorem 6.3. *Let $\Gamma \leq \mathrm{Out}(\mathbb{F})$ be purely atoroidal and convex cocompact, where $\mathbb{F}$ is a free group of finite rank at least 3, and let $\partial\iota\colon \partial\mathbb{F} \to \partial E_\Gamma$ denote the Cannon–Thurston map for the hyperbolic $\mathbb{F}$–extension E_Γ. Then for every $y \in \partial E_\Gamma$, the degree $\deg(y) = \#\left((\partial\iota)^{-1}(y)\right)$ of the fiber over y satisfies*

$$1 \leq \deg(y) \leq 2\,\mathrm{rank}(\mathbb{F}).$$

In particular, the fibers Cannon–Thurston map are all finite and of uniformly bounded size.

To establish Theorem 6.3, we relate Mitra's theory of "ending laminations" (Definition 4.3) for hyperbolic group extensions [Mit1] to the theory of algebraic laminations on free groups developed by Coulbois, Hilion, and Lustig [CHL3, CHL4]. A general result of Mitra [Mit1] about Cannon–Thurston maps for short exact sequences of hyperbolic groups (Theorem 4.6 below) implies that distinct points $p, q \in \partial\mathbb{F}$ are identified by the Cannon–Thurston map $\partial\iota\colon \partial\mathbb{F} \to \partial E_\Gamma$ if and only if (p, q) is a leaf of an ending lamination Λ_z on $\mathbb{F}$ for some $z \in \partial\Gamma$. Since our Γ is convex cocompact by assumption, the orbit map $\Gamma \to \mathcal{F}$ into the free factor complex extends to a continuous embedding $\partial\Gamma \to \partial\mathcal{F}$. By Bestvina–Reynolds [BR2] and Hamenstädt [Ham], the boundary $\partial\mathcal{F}$ consist of equivalence classes of arational $\mathbb{F}$–trees. Thus to each $z \in \partial\Gamma$ we may also associate a class T_z of arational $\mathbb{F}$–trees. The tree T_z moreover comes equipped with a *dual lamination* $L(T_z)$, as introduced in [CHL4]. Informally, $L(T_z)$ consists of lines in the free group which project to bounded diameter sets in the tree T_z. Our key technical result, Theorem 5.2, shows that for every $z \in \partial\Gamma$ we have $\Lambda_z = L(T_z)$. Combining this with Mitra's general theory [Mit1], we obtain the following explicit description of the Cannon–Thurston map:

Corollary 5.3. *Let $\Gamma \le \mathrm{Out}(\mathbb{F})$ be convex cocompact and purely atoroidal. Then the Cannon–Thurston map $\partial\iota\colon \partial\mathbb{F} \to \partial E_\Gamma$ identifies points $a, b \in \partial\mathbb{F}$ if and only if there exists $z \in \partial\Gamma$ such that $(a, b) \in L(T_z)$. That is, $\partial\iota$ factors through the quotient of $\partial\mathbb{F}$ by the equivalence relation*

$$a \sim b \iff (a, b) \in L(T_z) \text{ for some } z \in \partial\Gamma$$

and descends to an E_Γ–equivariant homeomorphism $\partial\mathbb{F}/\sim \to \partial E_\Gamma$.

We derive Theorem 6.3 from Corollary 5.3 by using results of Coulbois–Hilion [CH2] concerning the $\mathcal{Q}$*–index* for very small minimal actions of $\mathbb{F}$ on $\mathbb{R}$–trees. The point is that the laminations Λ_z that Mitra constructs in [Mit1] are a priori complicated and unwieldy objects from which it is difficult to extract information, whereas the laminations $L(T_z)$ appearing in Corollary 5.3 are subject to the general theory $\mathbb{R}$–trees. The equality $\Lambda_z = L(T_z)$ provided by Theorem 5.2 thus allows us to access this theory and use it to analyze the Cannon–Thurston maps.

However, establishing the equality $\Lambda_z = L(T_z)$ is nontrivial even in the special case, treated by Kapovich and Lustig [KL5], when $\Gamma = \langle\phi\rangle$ is a cyclic group generated by an atoroidal fully irreducible element ϕ. The general case considered here is considerably harder since our trees T_z no longer enjoy the "self-similarity" properties of stable trees of atoroidal fully irreducibles. The laminations Λ_z and $L(T_z)$ are defined in very different terms, and the main difficulty is in establishing the inclusion $\Lambda_z \subseteq L(T_z)$. The key step in this direction is Proposition 5.8 which shows that if $g_i \in \Gamma$ is a quasigeodesic sequence converging to $z \in \partial\Gamma$, then $\ell_{T_z}(g_i(h)) \to 0$ for every nontrivial $h \in \mathbb{F}$. Note that in this situation it is fairly straightforward to see that the projective geodesic current $[\mu] = \lim_{i\to\infty}[\eta_{g_i(h)}]$ satisfies $\langle T_z, \mu\rangle = 0$ (where $\langle\cdot,\cdot\rangle$ is the intersection pairing constructed in [KL1]), but this is much weaker than the needed conclusion $\lim_{i\to\infty} \ell_{T_z}(g_i(h)) = 0$. The proof of Proposition 5.8 relies on recent results of Dowdall and Taylor [DT1] about folding paths in Culler and Vogtmann's Outer space $\mathfrak{X}$ that remain close to the orbit of a purely atoroidal convex cocompact subgroup Γ.

Our Theorem 5.2, establishing that for every $z \in \partial\Gamma$ we have $\Lambda_z = L(T_z)$, has quickly found useful applications in a new paper of Mj and Rafi [MR2] regarding quasiconvexity in the context of hyperbolic group extensions. See Proposition 4.3 in [MR2] and its applications in Theorem 4.11 and Theorem 4.12 of [MR2]. We remark that the quasiconvexity result given by Theorem 4.12 of [MR2] is also proved by different methods in the forthcoming paper [DT2].

Rational and essential points. In addition to Theorem 6.3 and Corollary 5.3, we obtain fine information about the Cannon–Thurston map in regards to rational and essential points. Recall that a point ξ in the boundary ∂G of a word-hyperbolic group G is called *rational* if there is an infinite-order element $g \in G$ such that ξ equals the limit g^∞ in $G \cup \partial G$ of the sequence $\{g^n\}$. For the short exact sequence (2), a point $y \in \partial E_\Gamma$ is called Γ*–essential* if there exists a (necessarily unique) point $\zeta(y) \in \partial\Gamma$ such that $y = \partial\iota(p)$ for a point $p \in \partial\mathbb{F}$ that is *proximal* for $L(T_{\zeta(y)})$ in the sense of Definition 3.9 below. Informally, Γ–essential points are the $\partial\iota$–images of points in $\partial\mathbb{F}$ that "remember" in an essential way the lamination $L(T_z)$ for some $z \in \partial\Gamma$.

We write $\deg(y) := \#\left((\partial\iota)^{-1}(y)\right)$ for the cardinality of the Cannon–Thurston fiber over $y \in \partial E_\Gamma$ (so $1 \le \deg(y) \le 2\,\mathrm{rank}(\mathbb{F})$ by Theorem 6.3). Every $y \in \partial E_\Gamma$ with $\deg(y) \ge 2$ is Γ–essential and moreover has Cannon–Thurston fiber given by $(\partial\iota)^{-1}(y) = \{p\} \cup \{q \in \partial\mathbb{F} \mid (p, q) \in L(T_{\zeta(y)})\}$ for *every* $p \in (\partial\iota)^{-1}(y)$ (Lemma 6.1). However, there are also may Γ–essential points with $\deg(y) = 1$. Our next result describes the fibers of $\partial\iota$ over rational points of ∂E_Γ.

Theorem 6.4. *Suppose that $1 \to \mathbb{F} \to E_\Gamma \to \Gamma \to 1$ is a hyperbolic extension with $\Gamma \le \mathrm{Out}(\mathbb{F})$ convex cocompact. Consider a rational point $g^\infty \in \partial E_\Gamma$, where $g \in E_\Gamma$ has infinite order.*

1. *Suppose that g^k is equal to $w \in \mathbb{F} \lhd E_\Gamma$ for some $k \ge 1$ (i.e., g projects to a finite order element of Γ). Then $(\partial\iota)^{-1}(g^\infty) = \{w^\infty\} \subset \partial\mathbb{F}$ and so $\deg(g^\infty) = 1$.*
2. *Suppose that g projects to an infinite-order element $\phi \in \Gamma$. Then there exists $k \ge 1$ such that the automorphism $\Psi \in \mathrm{Aut}(\mathbb{F})$ given by $\Psi(w) = g^k w g^{-k}$ is forward rotationless (in the sense of [FH, CH1]) and its set $\mathrm{att}(\Psi)$ of attracting fixed points in $\partial\mathbb{F}$ is exactly $\mathrm{att}(\Psi) = (\partial\iota)^{-1}(g^\infty)$. Moreover, g^∞ is Γ–essential and $\zeta(g^\infty) = \phi^\infty$.*

In the case of a cyclic group $\langle\phi\rangle$ generated by an atoroidal fully irreducible automorphism ϕ, Kapovich and Lustig [KL5] showed that every point $y \in \partial E_{\langle\phi\rangle}$ with $\deg(y) \ge 3$ is rational. We show that when Γ is nonelementary this conclusion no longer holds and rather that, with some unavoidable exceptions, rational points in ∂E_Γ come in a specific way from rational points in $\partial\Gamma$:

Theorem 6.5. *Suppose that $1 \to \mathbb{F} \to E_\Gamma \to \Gamma \to 1$ is a hyperbolic extension with $\Gamma \le \mathrm{Out}(\mathbb{F})$ convex cocompact. Then the following hold:*

1. *If $y \in \partial E_\Gamma$ has $\deg(y) \ge 3$ and $\zeta(y) \in \partial\Gamma$ is rational, then y is rational.*
2. *If $y \in \partial E_\Gamma$ has $\deg(y) \ge 2$ and $\zeta(y) \in \partial\Gamma$ is irrational, then y is irrational.*

Conical limit points. Recall that a point ξ in the boundary ∂G of a word-hyperbolic group G is a *conical limit point* for the action of a subgroup $H \le G$ on ∂G if there exists a geodesic ray in the Cayley graph of G that converges to ξ and has a bounded neighborhood that contains infinitely many elements of H. Combining the results of this paper with the results of [JKLO], we obtain the following:

Theorem 6.6. *Let $\Gamma \le \mathrm{Out}(\mathbb{F})$ be purely atoroidal and convex cocompact. If $y \in \partial E_\Gamma$ is Γ–essential, then y is not a conical limit point for the action of $\mathbb{F}$ on ∂E_Γ. In particular, if $\deg(y) \ge 2$ or if $y = g^\infty$ for some $g \in E_\Gamma$ projecting to an infinite-order element of Γ, then y is not a conical limit point for the action of $\mathbb{F}$.*

It is known (see [Ger, JKLO]) in a very general convergence group situation that if a Cannon–Thurston map exists then every conical limit point has exactly one pre-image under the Cannon–Thurston map; thus points with ≥ 2 pre-images cannot be conical limit points. However, Theorem 6.6 also applies to many Γ–essential points $y \in \partial E_\Gamma$ with $\deg(y) = 1$.

Discontinuity of ending laminations. In [Mit1], Mitra asks whether the map which associates to each point $z \in \partial\Gamma$ the corresponding ending lamination Λ_z is continuous with respect to the Chabauty topology on the space of laminations. Of course, in the case of extensions by $\mathbb{Z}$ there is nothing to check since the boundary $\partial\mathbb{Z}$ is discrete. In Section 7, we answer Mitra's question in the negative by producing a hyperbolic extension E_Γ for which the map $z \mapsto \Lambda_z$ is not continuous. This is done explicitly in Example 7.5.

Besides establishing this discontinuity, we also provide a positive result about subconvergence of ending laminations. For the statement, let $\mathcal{L}(\mathbb{F})$ denote the space of laminations on $\mathbb{F}$ equipped with the Chabauty topology (recalled in Definition 3.1) and let Λ_z denote Mitra's [Mit1] ending lamination for $z \in \partial\Gamma$ (see Definition 4.3). For a lamination $L \in \mathcal{L}(\mathbb{F})$, the notation L' denotes the set of accumulation points of L, in the usual topological sense.

Proposition 7.1. *Let $\Gamma \le \mathrm{Out}(\mathbb{F})$ be purely atoroidal and convex cocompact, and let $\Lambda_z \in \mathcal{L}(\mathbb{F})$ denote the ending lamination associated to $z \in \partial\Gamma$. Then for any sequence z_i in $\partial\Gamma$ converging to z and any subsequence limit L of the corresponding sequence Λ_{z_i} in $\mathcal{L}(\mathbb{F})$, we have*

$$\Lambda'_z \subset L \subset \Lambda_z.$$

This result can be viewed as a statement about the map $\partial\Gamma \to \mathcal{L}(\mathbb{F})$, given by $z \mapsto \Lambda_z$, possessing a weak form of continuity.

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2. Cannon–Thurston maps

In this section, we recall some facts about Cannon–Thurston maps for general hyperbolic extensions. For a word-hyperbolic group G, we denote its Gromov boundary by ∂G. The following result establishes the existence of the Cannon–Thurston map:

Proposition-Definition 2.1 (Mitra [Mit2])**.** *Suppose that $1 \to H \to G \to \Gamma \to 1$ is an exact sequence of word-hyperbolic groups. Then the inclusion $\iota\colon H \to G$ admits a continuous extension $\hat{\iota}\colon H \cup \partial H \to G \cup \partial G$ with $\hat{\iota}(\partial H) \subseteq \partial G$. The restricted map $\partial\iota := \hat{\iota}|_{\partial H}\colon \partial H \to \partial G$ is called the* Cannon–Thurston map *for the inclusion $H \to G$; it is surjective whenever H is infinite.*

For an element $g \in G$ denote by Φ_g the automorphism $h \mapsto ghg^{-1}$ of H. We denote by $\phi_g \in \mathrm{Out}(H)$ the outer automorphism class of Φ_g. Similarly, for an element $q \in \Gamma$ denote by $\phi_q \in \mathrm{Out}(H)$ the outer automorphism class of Φ_g, where $g \in G$ is any element that maps to q; note that the class ϕ_q is independent of the chosen lift g. For a conjugacy class $[h]$ in H and an element $q \in \Gamma$ we also write $[q(h)] := [\phi_q(h)] = [ghg^{-1}]$, where $g \in G$ is any element projecting to q.

By construction, the Cannon–Thurston map $\partial\iota\colon \partial H \to \partial G$ in Proposition-Definition 2.1 is H–equivariant with respect to the left translation actions of H on ∂H and ∂G. However, $\partial\iota$ actually turns out to be G–equivariant with respect to the action $G \curvearrowright \partial H$ defined by $g \cdot p := \Phi_g(p)$ for $g \in G$ and $p \in \partial H$. Notice that the restricted action $H \curvearrowright \partial H$, namely $h \cdot p = \Phi_h(p)$, agrees with the usual action of H on ∂H by left translation.

Proposition 2.2. *Suppose $1 \to H \to G \to \Gamma \to 1$ is an exact sequence of hyperbolic groups. Then the map $(g, p) \mapsto g \cdot p$ defines an action of G on ∂H by homeomorphisms. Moreover, the Cannon–Thurston map $\partial\iota\colon \partial H \to \partial G$ is G–equivariant.*

Proof. While this is implicit in [KL5], we include a proof for completeness. The fact that $(g, p) \mapsto g \cdot p$ defines a group action by homeomorphisms follows directly from the definitions. Choose $p \in \partial H$ and $g \in G$. To prove G–equivariance we must show that $\partial\iota(\Phi_g(p)) = g \cdot \partial\iota(p)$.

Choose a sequence $h_n \in H$ such that $h_n \to p$ in the topology of $H \cup \partial H$. By definition of $\partial\iota$ it follows that $h_n \to \partial\iota(p)$ in the topology of $G \cup \partial G$. In G we have $gh_ng^{-1} = \Phi_g(h_n)$ so that

$$g \cdot \partial\iota(p) = \lim_{n\to\infty} gh_n = \lim_{n\to\infty} gh_ng^{-1} = \lim_{n\to\infty} \Phi_g(h_n)$$

in the topology of $G \cup \partial G$. But definition of $\partial\iota$ the last limit above is exactly $\partial\iota(\Phi_g(p))$. □

3. Background on free groups, laminations and $\mathcal{Q}$–index

For the entirety of this section let $\mathbb{F}$ be a free group of finite rank $N \geq 2$. We will also fix a free basis X of $\mathbb{F}$ and the Cayley graph $\mathrm{Cay}(\mathbb{F}, X)$ of $\mathbb{F}$ with respect to X.

3.1. Laminations on free groups.

We denote $\partial^2\mathbb{F} := \{(p, q) \in \partial\mathbb{F} \times \partial\mathbb{F} \colon p \neq q\}$ and endow $\partial^2\mathbb{F}$ with the subspace topology from the product topology on $\partial\mathbb{F} \times \partial\mathbb{F}$. There is a natural diagonal left action of $\mathbb{F}$ on $\partial^2\mathbb{F}$ by left translations: $w(p, q) := (wp, wq)$ where $w \in \mathbb{F}$ and $(p, q) \in \partial^2\mathbb{F}$.

An *algebraic lamination* on $\mathbb{F}$ is a closed $\mathbb{F}$–invariant subset $L \subseteq \partial^2\mathbb{F}$ such that L is also invariant with respect to the "flip map" $\partial^2\mathbb{F} \to \partial^2\mathbb{F}$, $(p, q) \mapsto (q, p)$. If L is an algebraic lamination on $\mathbb{F}$, an element $(p, q) \in L$ is also referred to as a *leaf* of L. In the Cayley graph $\mathrm{Cay}(\mathbb{F}, X)$ of $\mathbb{F}$ with respect to the free basis X, every leaf $(p, q) \in L$ is represented by a unique unparameterized bi-infinite geodesic l from p to q in $\mathrm{Cay}(\mathbb{F}, X)$. In this situation we will also sometimes say that l is a leaf of L. We refer the reader to [CHL3, CHL4] for the background information on algebraic laminations.

We say that a subset $L \subseteq \partial^2\mathbb{F}$ is *diagonally closed* if whenever $p, q, r \in \partial\mathbb{F}$ are three distinct points such that $(p, q), (q, r) \in L$ then $(p, r) \in L$.

An important class of laminations are those corresponding to conjugacy classes of $\mathbb{F}$. For $g \in \mathbb{F} \setminus \{1\}$, we denote by $g^{+\infty} \in \partial\mathbb{F}$ the unique forward limit of the sequence $(g^n)_{n\geq 1}$ in $\mathbb{F} \cup \partial\mathbb{F}$. Define $g^{-\infty}$ similarly and note that $(g^{-1})^{+\infty} = g^{-\infty}$. Then define the algebraic lamination

$$L(g) = \mathbb{F} \cdot (g^{+\infty}, g^{-\infty}) \cup \mathbb{F} \cdot (g^{-\infty}, g^{\infty}).$$

Note that $L(g)$ depends only on the conjugacy class of g. Moreover, $L(g)$ is indeed a closed subset of $\partial^2\mathbb{F}$ and so is a bona fide algebraic lamination. In what follows, for a subset A of a topological space, we denote the closure of A by $\overline{A}$ and its set of accumulation points by A'. For a collection Ω of conjugacy classes of $\mathbb{F}$, we let $L(\Omega)$ denote the smallest algebraic lamination containing $L(g)$ for each $g \in \Omega$. We observe that

$$L(\Omega) = \overline{\bigcup_{g\in\Omega} L(g)} \tag{3}$$

Since each $L(g)$ is itself closed in $\partial^2\mathbb{F}$, we see that the above closure is unnecessary when Ω is finite.

Finally, we denote the set of all laminations of $\mathbb{F}$ by $\mathcal{L}(\mathbb{F})$, which we consider with the Chabauty topology. We recall the definition of this topology:

Definition 3.1 (Topology on $\mathcal{L}(\mathbb{F})$)**.** Let Y be a locally compact metric space and let $C(Y)$ be the collection of closed subsets of Y. The *Chabauty topology* on $C(Y)$ is defined as the topology generated by the subbasis consisting of

(1) $\mathcal{U}_1(K) = \{C \in C(Y) : C \cap K = \emptyset\}$ for $K \subset Y$ compact.
(2) $\mathcal{U}_2(O) = \{C \in C(Y) : C \cap O \neq \emptyset\}$ for $O \subset Y$ open.

A geometric interpretation of convergence in the Chabauty topology is stated in Lemma 7.2; it will be needed in Section 7. Recall that the space $C(Y)$ is always compact [CME]. Returning to the situation of algebraic laminations of $\mathbb{F}$, we note that $\mathcal{L}(\mathbb{F})$ is closed in $C(\partial^2\mathbb{F})$ and hence is itself compact. We henceforth consider $\mathcal{L}(\mathbb{F})$ with the subspace topology and refer to this as the Chabauty topology on $\mathcal{L}(\mathbb{F})$.

3.2. **Outer space and its boundary.** Outer space, denoted cv and introduced by Culler–Vogtmann in [CV], is the space of $\mathbb{F}$–marked metric graphs, up to some natural equivalence. A *marked graph* (G, ϕ) is a core graph G (finite with no valence one vertices) equipped with a marking $\phi\colon R \to G$, which is a homotopy equivalence from a fixed rose R with rank($\mathbb{F}$) petals to the graph G. A *metric* on G is a function ℓ assigning to each edge of G a positive real number (its *length*) and we call the sum of the lengths of the edges of G its *volume*. A marked metric graph is a triple (G, ϕ, ℓ), and Outer space is defined to be set of marked metric graph up to *equivalence*, where (G_1, ϕ_1, ℓ_1) is equivalent to (G_2, ϕ_2, ℓ_2) if there is an isometry from G_1 to G_2 in the homotopy class of the *change of marking* $\phi_2 \circ \phi_1^{-1}\colon G_1 \to G_2$. *Projectivized Outer space* $\mathfrak{X}$, also sometimes denoted CV, is then defined to be the subset of cv consisting of graphs of volume 1. Although points in cv are as described above, we will often denote a marked metric graph simply by its underlying graph G suppressing the marking and metric.

Given $G \in$ cv, the marking associated to G allows one to measure the length of a conjugacy class α of $\mathbb{F}$. In particular, there is a unique immersed loop in G corresponding to the homotopy class α which we denote by $\alpha|G$. The *length of* α *in* G, denoted $\ell(\alpha|G)$, is the sum of the lengths of the edges of G crossed by $\alpha|G$, counted with multiplicites. The standard topology on cv is defined as the smallest topology such that each of the length functions $\ell(\alpha|\cdot)\colon \text{cv} \to \mathbb{R}_+$ is continuous [CV, Pau].

Given a point (G, ϕ, ℓ) in cv, we can define T to be the universal cover of G equipped with a metric obtained by lifting the metric ℓ and also equipped with an action of $\mathbb{F}$ on T by covering transformations (where $\mathbb{F}$ and $\pi_1(G)$ are identified via the marking ϕ). Then T is an $\mathbb{R}$–tree equipped with a minimal free discrete isometric action of $\mathbb{F}$. Under this correspondence, equivalent marked metric graphs correspond to $\mathbb{F}$–equivariantly isometric $\mathbb{R}$–trees. This procedure provides an identification between cv and the space of minimal free discrete isometric actions of $\mathbb{F}$ on $\mathbb{R}$–trees, considered up to $\mathbb{F}$–equivariant isometries. If T corresponds to (G, ϕ, ℓ) then for every $w \in \mathbb{F}$ we have $\ell(w|G) = \ell_T(w) := \min_{x \in T} d_T(x, wx)$. We will also sometime use the notation $i(T, w)$ to denote the translation length of w in T, i.e. $i(T, w) = \ell_T(w)$. This notation refers to the intersection form studied in [KL1]; the details of which are not needed here.

We denote by $\overline{\text{cv}}$ the set of all very small minimal isometric actions of $\mathbb{F}$ on $\mathbb{R}$–trees, considered up to $\mathbb{F}$–equivariant isometries. As usual, for $T \in \overline{\text{cv}}$ and $w \in \mathbb{F}$, define the *translation length* of w on T as $\ell_T(w) := \inf_{x \in T} d(x, wx)$. It is known that $\overline{\text{cv}}$ is equal to the closure of cv with respect to the "axes topology;" see [CL, BF1] for the original proof, and see [Gui1] for a generalization. We denote the projectivization of $\overline{\text{cv}}$ by $\overline{\mathcal{X}} = \mathcal{X} \cup \partial\mathcal{X}$. Hence, $\partial\mathcal{X}$ denotes projective classes of very small minimal actions of $\mathbb{F}$ on $\mathbb{R}$–trees which are not free and simplicial; this is the so-called boundary of Outer space. We remark that $\overline{\mathcal{X}}$ is compact.

We recall how Aut($\mathbb{F}$) and Out($\mathbb{F}$) act on $\overline{\text{cv}}$. If $T \in \overline{\text{cv}}$ and $\Phi \in \text{Aut}(\mathbb{F})$, the tree $T\Phi \in \overline{\text{cv}}$ is defined as follows. As a set and a metric space we have $T\Phi = T$. The action of $\mathbb{F}$ is modified via Φ: for every $x \in T$ and $w \in \mathbb{F}$ we have $w \underset{T\Phi}{\cdot} x = \Phi(w) \underset{T}{\cdot} x$. This formula defines a right action of Aut($\mathbb{F}$) on $\overline{\text{cv}}$. The subgroup $\text{Inn}(\mathbb{F}) \leq \text{Aut}(\mathbb{F})$ is contained in the kernel of this action and therefore the action descends to a right action of Out($\mathbb{F}$) on $\overline{\text{cv}}$: for $\phi \in \text{Out}(\mathbb{F})$ and $T \in \overline{\text{cv}}$ we have $T\phi := T\Phi$, where $\Phi \in \text{Aut}(\mathbb{F})$ is any automorphism in the outer automorphism class ϕ. At the level of translation length functions, for $T \in \overline{\text{cv}}$, $w \in \mathbb{F}$ and $\phi \in \text{Out}(\mathbb{F})$ we have $\ell_{T\phi}(w) = \ell_T(\phi(w))$. Finally, these right actions of Aut($\mathbb{F}$) and Out($\mathbb{F}$) on $\overline{\text{cv}}$ can be transformed into left actions by putting $\Phi T := T\Phi^{-1}$, $\phi T := T\phi^{-1}$ for $T \in \overline{\text{cv}}$, $\phi \in \text{Out}(\mathbb{F})$ and $\Phi \in \text{Aut}(\mathbb{F})$.

3.3. **Metric properties of Outer space.** For the applications in this paper, we will need a few facts from the metric theory of Outer space. We refer the reader to [FM, BF2, DT1] for details on the relevant background.

If $T_1 = (G_1, \phi_1, \ell_1)$ and $T_2 = (G_2, \phi_2, \ell_2)$ are two points in cv, the *extremal Lipschitz distortion* $\text{Lip}(T_1, T_2)$, also sometimes denoted $\text{Lip}(G_1, G_2)$, is the infimum of the Lipschitz constants of all the Lipschitz maps $f \colon (G_1, \ell_1) \to (G_2, \ell_2)$ that are freely homotopic to the the change of marking $\phi_2 \circ \phi_1^{-1}$. If one views T_1 and T_2 as $\mathbb{R}$–trees, then $\text{Lip}(T_1, T_2)$ is the infimum of the Lipschitz constants among all $\mathbb{F}$–equivariant Lipschitz maps $T_1 \to T_2$. It is known that

$$\text{Lip}(T_1, T_2) = \max_{w \in \mathbb{F} \setminus \{1\}} \frac{\ell_{T_2}(w)}{\ell_{T_1}(w)}.$$

For $T_1, T_2 \in \mathcal{X}$ we put

$$d_{\mathcal{X}}(T_1, T_2) := \log \text{Lip}(T_1, T_2)$$

and call $d_{\mathcal{X}}(T_1, T_2)$ the *asymmetric Lipschitz distance* from T_1 to T_2. It is known that $d_{\mathcal{X}}$ satisfies all the axioms of being a metric on $\mathcal{X}$ except that $d_{\mathcal{X}}$ is, in general, not symmetric as there exist $T_1, T_2 \in \mathcal{X}$ such that $d_{\mathcal{X}}(T_1, T_2) \neq d_{\mathcal{X}}(T_2, T_1)$. Because of this asymmetry, it is sometimes convenient to consider the symmetrization of the Lipschitz metric:

$$d_{\mathcal{X}}^{\text{sym}}(T_1, T_2) := d_{\mathcal{X}}(T_1, T_2) + d_{\mathcal{X}}(T_2, T_1)$$

which is an actual metric on $\mathcal{X}$ and induces the standard topology [FM]. For a subset $A \in \mathcal{X}$, we denote by $N_K(A)$ the *symmetric K–neighborhood* of A, which is the neighborhood of A considered with the symmetric metric.

It is known that for any $T_1, T_2 \in \mathcal{X}$ there exists a unit-speed $d_{\mathcal{X}}$–geodesic $\gamma \colon [a, b] \to \mathcal{X}$ given by a *standard geodesic* from T_1 to T_2 in $\mathcal{X}$. Such a geodesic is a concatenation of a *rescaling path*, which only alters the edge lengths of T_1, followed by a *folding path*. This geodesic has the property that $\gamma(a) = T_1$, $\gamma(b) = T_2$, $b - a = d_{\mathcal{X}}(T_1, T_2)$ and that for any $a \leq t \leq t' \leq b$ one has $t' - t = d_{\mathcal{X}}(\gamma(t), \gamma(t'))$. The folding path $\gamma(s)$ has some additional properties arising from its specific construction. We omit describing these properties for the moment (and refer the reader to [FM, BF2, DT1] for details), but will use them as needed in our arguments.

If $\mathbf{I} \subseteq \mathbb{R}$ is a (possibly infinite) interval, we also say that $\gamma\colon \mathbf{I} \to \mathcal{X}$ is a folding path if for every finite subinterval $[a,b] \subseteq \mathbf{I}$ the restriction $\gamma|_{[a,b]}\colon [a,b] \to \mathcal{X}$ is a folding path giving a unit speed $d_{\mathcal{X}}$–geodesic in the above sense. Then $\gamma\colon \mathbf{I} \to \mathcal{X}$ is also a unit-speed $d_{\mathcal{X}}$–geodesic.

3.4. Dual laminations of very small trees.

Definition 3.2 (Dual lamination)**.** Let $T \in \overline{\mathrm{cv}}$. For each $\epsilon > 0$, let $\Omega^{\le\epsilon}(T)$ denote the collection of $1 \neq g \in \mathbb{F}$ with $\ell_T(g) \le \epsilon$. We form the algebraic lamination generated by these ϵ–short conjugacy classes:

$$L^{\le\epsilon}(T) = L(\Omega^{\le\epsilon}(T)) = \overline{\bigcup_{g\in\Omega^{\le\epsilon}(T)} L(g)} \subset \partial^2\mathbb{F}.$$

The *dual lamination* $L(T) \subseteq \partial^2\mathbb{F}$ of T is then defined to be

$$L(T) := \bigcap_{\epsilon>0} L^{\le\epsilon}(T).$$

Remark 3.3. Note that $L^{\le\epsilon}(T)$ and $L(T)$ are in fact algebraic laminations on $\mathbb{F}$. Further, it is well-known [CHL4] and not hard to show that $L(T)$ consists of all $(p,q) \in \partial^2\mathbb{F}$ such that for every $\epsilon > 0$ and every finite subword v of the bi-infinite geodesic from p to q in $\mathrm{Cay}(\mathbb{F}, X)$ there exists a cyclically reduced word w over $X^{\pm1}$ with $\ell_T(w) \le \epsilon$ such that v is a subword of w.

In this paper, we will only be concerned with a certain class of trees $T \in \overline{\mathrm{cv}}$:

Definition 3.4 (Arational tree)**.** A tree $T \in \overline{\mathrm{cv}}$ is called *arational* if there does not exist a proper free factor F of $\mathbb{F}$ and an F–invariant subtree $Y \subseteq T$ such that F acts on Y with dense orbits.

In [Rey] Reynolds obtained a useful characterization of arational trees in different terms. This characterization implies that if $T \in \overline{\mathrm{cv}}$ does not arise as a dual tree to a geodesic lamination on a once-punctured surface, then T is arational if and only if T is "indecomposable" (in the sense of [Gui2]) and $\mathbb{F}$ acts on T freely with dense orbits. In particular, if $\phi \in \mathrm{Out}(\mathbb{F})$ is an atoroidal fully irreducible, then the stable tree T_ϕ (discussed in Section 3.7 below) is free and arational; see [CH1].

3.5. The factor complex and its boundary.

The *free factor complex* of $\mathbb{F}$ (for $\mathrm{rank}(\mathbb{F}) \ge 3$) is the complex $\mathcal{F}$ defined as following: vertices of $\mathcal{F}$, are conjugacy classes of free factors of $\mathbb{F}$ and vertices $A_0, \ldots, A_k$ span an k–simplex if these classes have nested representatives $A_0 < \cdots < A_k$. The complex $\mathcal{F}$ was introduced in [HV] and has since become a central tool for studying the geometry of $\mathrm{Out}(\mathbb{F})$. In particular, the following theorem is most important for our purposes.

Theorem 3.5 (Bestvina–Feighn [BF2])**.** *The free factor complex $\mathcal{F}$ is Gromov-hyperbolic; moreover, an element $\phi \in \mathrm{Out}(\mathbb{F})$ acts on $\mathcal{F}$ as a loxodromic isometry if and only if ϕ is fully irreducible.*

A central tool in the proof of Theorem 3.5 is the coarse Lipschitz projection $\pi\colon \mathcal{X} \to \mathcal{F}$ from Outer space to the factor complex, which is defined by sending $G \in \mathcal{X}$ to the collection

$$\pi(G) = \{\pi_1(G') : G' \subset G \text{ is a connected, proper subgraph}\} \subset \mathcal{F}^0.$$

By [BF2, Lemma 3.1], $\mathrm{diam}_{\mathcal{F}}(\pi(G)) \le 4$. Further, there is an $L \ge 0$, depending only on $\mathrm{rank}(\mathbb{F})$, such that $\pi\colon \mathcal{X} \to \mathcal{F}$ is coarsely L–Lipschitz [BF2, Corollary 3.5]. Moreover [BF2, Theorem 9.3], if $\gamma\colon [a,b] \to \mathcal{X}$ is a folding path, then $\pi(\gamma([a,b]))$ is within a uniform Hausdorff distance (independent of γ) from any $\mathcal{F}$–geodesic from $\pi(\gamma(a))$ to $\pi(\gamma(b))$.

As a hyperbolic space, $\mathcal{F}$ has a Gromov boundary. Let $\mathcal{AT}$ be the subspace of $\partial\mathcal{X}$ consisting of projective classes of arational trees. For $T, T' \in \mathcal{AT}$, define $T \approx T'$ to mean that $L(T) = L(T')$. Thus $\approx$ is an equivalence relation on $\mathcal{AT}$. The following theorem computes the boundary of $\mathcal{F}$ and will be needed in Section 5.

Theorem 3.6 (Bestvina–Reynolds [BR2], Hamenstädt [Ham])**.** *The projection $\pi\colon \mathcal{X} \to \mathcal{F}$ has an extension to a map $\partial\pi\colon \mathcal{AT} \to \partial\mathcal{F}$ which satisfies the following properties:*

- *If $(G_i)_{i\ge0} \subset \mathcal{X}$ is a sequence converging in $\overline{\mathcal{X}}$ to $T \in \mathcal{AT}$, then $\pi(G_i) \to \partial\pi(T)$ in $\mathcal{F} \cup \partial\mathcal{F}$.*

- *If $(G_i)_{i\geq 0} \subset \mathcal{X}$ is a sequence converging in $\overline{\mathcal{X}}$ to $T \in \overline{\mathcal{X}} \setminus \mathcal{AT}$, then the sequence $(\pi(G_i))_{i\geq 0}$ remains bounded in $\mathcal{F}$.*

Moreover, if $T \approx T'$ then $\partial\pi(T) = \partial\pi(T')$, and the induced map $(\mathcal{AT}/\approx) \to \partial F$ is a homeomorphism.

We also record the following useful statement which follows directly from [CHR, Theorem A]:

Proposition 3.7. *Let $T, T' \in \overline{cv}$ be free arational trees such that $T \not\approx T'$. Then $L(T) \cap L(T') = \varnothing$. Moreover if $(p, q) \in L(T)$ then there does not exist $q' \in \partial\mathbb{F}$ such that $(p, q') \in L(T')$.*

3.6. **The $\mathcal{Q}$–map and the $\mathcal{Q}$–index.** For a tree $T \in \overline{\mathrm{cv}}$, denote $\hat{T} := \overline{T} \cup \partial T$, where $\overline{T}$ is the metric completion of T and ∂T is the hyperbolic boundary of T. Note that the action of $\mathbb{F}$ on T naturally extends to an action of $\mathbb{F}$ on $\hat{T}$.

For a tree $T \in \overline{\mathrm{cv}}$ with dense $\mathbb{F}$–orbits, Coulbois, Hilion and Lustig [CHL4] constructed an $\mathbb{F}$–equivariant surjective map $\mathcal{Q}_T \colon \partial\mathbb{F} \to \hat{T}$. The precise definition of $\mathcal{Q}_T$ is not important for our purposes but we will need the following crucial property of $\mathcal{Q}_T$:

Proposition 3.8. [CHL4, Proposition 8.5] *Let $T \in \overline{cv}$ be a tree with dense $\mathbb{F}$–orbits. Then for distinct points $p, p' \in \partial\mathbb{F}$ we have $\mathcal{Q}_T(p) = \mathcal{Q}_T(p')$ if and only if $(p, p') \in L(T)$.*

For a tree $T \in \overline{\mathrm{cv}}$ we say that a freely reduced word v over $X^{\pm 1}$ is an *X–leaf segment* for $L(T)$ (or just a *leaf segment* for $L(T)$) if there exists $(p, p') \in L(T)$ such that v labels a finite subpath of the bi-infinite geodesic from p to p' in $\mathrm{Cay}(\mathbb{F}, X)$.

Definition 3.9. A point $p \in \partial\mathbb{F}$ is said to be *proximal* for $L(T)$ if for every v such that v occurs infinitely often as a subword of the geodesic ray from 1 to p in $\mathrm{Cay}(\mathbb{F}, X)$, the word v is a leaf segment for $L(T)$.

Proposition 3.10. *The following hold:*

1. *For $T \in \overline{cv}$ the definition of a proximal points for $L(T)$ does not depend on the free basis X.*
2. *If $T, T' \in \overline{cv}$ are free arational trees such that there exists a point $p \in \partial\mathbb{F}$ that is proximal for both $L(T)$ and $L(T')$, then $L(T) = L(T')$.*

Proof. Part (1) easily follows from the fact that for any two free bases X_1, X_2 of $\mathbb{F}$, the identity map $\mathbb{F} \to \mathbb{F}$ extends to a quasi-isometry $\mathrm{Cay}(\mathbb{F}, X_1) \to \mathrm{Cay}(\mathbb{F}, X_2)$. We leave the details to the reader.

For part (2), suppose that $T, T' \in \overline{\mathrm{cv}}$ are free arational trees such that there exists $p \in \partial\mathbb{F}$ which is proximal for both $L(T)$ and $L(T')$. Therefore for every $n \geq 1$ there exists a freely reduced word over $X^{\pm 1}$ of length n which is a leaf-segment for both $L(T)$ and $L(T')$. By a standard compactness argument it then follows that there exists a point $(p_1, q_1) \in L(T) \cap L(T')$. Therefore by Proposition 3.7 we have $L(T) = L(T')$. □

We will need the following known results about the map $\mathcal{Q}_T$:

Proposition 3.11. *Let $T \in \overline{cv}$ be a free $\mathbb{F}$–tree with dense $\mathbb{F}$–orbits. Then the following hold:*

1. [CHL4, Proposition 5.8] *If $p \in \partial\mathbb{F}$ is such that $\mathcal{Q}_T(p) \in \overline{T}$, then p is proximal for $L(T)$.*
2. [CH2, Proposition 5.2] *For every $x \in \partial T$ we have $\#(\mathcal{Q}_T^{-1}(x)) = 1$.*
3. *For every $x \in \hat{T}$ we have $1 \leq \#(\mathcal{Q}_T^{-1}(x)) < \infty$.*
4. *There are only finitely many $\mathbb{F}$–orbits of points $x \in \hat{T}$ with $\#(\mathcal{Q}_T^{-1}(x)) \geq 3$.*

Associated to the map $\mathcal{Q}_T$ there is a notion of the $\mathcal{Q}$–index of T, developed in [CH2]. We will only need the definition and properties of the $\mathcal{Q}$–index for the case where $T \in \overline{\mathrm{cv}}$ is a free $\mathbb{F}$–tree with dense orbits, and so we restrict our consideration to that context.

Definition 3.12 ($\mathcal{Q}$–index)**.** Let $T \in \overline{\mathrm{cv}}$ be a free $\mathbb{F}$–tree with dense $\mathbb{F}$–orbits. The *$\mathcal{Q}$–index* of a point $x \in \hat{T}$ is defined to be $\mathrm{ind}_{\mathcal{Q}}(x) := \max\{0, -2 + \#(\mathcal{Q}_T^{-1}(x))\}$. The *$\mathcal{Q}$–index* of the tree T is then defined as

$$\mathrm{ind}_{\mathcal{Q}}(T) := \sum \mathrm{ind}_{\mathcal{Q}}(x),$$

where the summation is taken over the set of representatives of $\mathbb{F}$–orbits of points x of $\hat{T}$ with $\#(\mathcal{Q}_T^{-1}(x)) \geq 3$.

The main result of [CH2] is the following:

Theorem 3.13. [CH2, Theorem 5.3] *Let $\mathbb{F}$ be a finite-rank free group with* $\mathrm{rank}(\mathbb{F}) \geq 3$*. Then every free $\mathbb{F}$–tree $T \in \overline{cv}$ with dense $\mathbb{F}$–orbits satisfies*

$$\mathrm{ind}_{\mathcal{Q}}(T) \leq 2\,\mathrm{rank}(\mathbb{F}) - 2.$$

3.7. **Stable trees of fully irreducibles.** For any fully irreducible $\phi \in \mathrm{Out}(\mathbb{F})$ there is an associated *stable tree* $T_\phi \in \overline{\mathrm{cv}}$ with the property that $\phi T_\phi = \lambda T_\phi$ for some $\lambda > 1$. The tree $T_\phi \in \overline{\mathrm{cv}}$ is uniquely determined by ϕ, up to multiplying the metric by a positive scalar, and the projective class $[T_\phi] \in \overline{\mathfrak{X}}$ is the unique attracting fixed point for the left action of ϕ on $\overline{\mathfrak{X}}$. The tree T_ϕ may be explicitly constructed from a train-track representative $f\colon G \to G$ of ϕ^{-1}, and the "eigenvalue" λ in the equation $\phi T_\phi = \lambda T_\phi$ is the Perron-Frobenius eigenvalue of the transition matrix of f. For any fully irreducible $\phi \in \mathrm{Out}(\mathbb{F})$ the tree T_ϕ has dense $\mathbb{F}$–orbits; if in addition ϕ is atoroidal then the action of $\mathbb{F}$ on T_ϕ is free.

Suppose now that $\phi \in \mathrm{Out}(\mathbb{F})$ is an atoroidal fully irreducible element, so that $\phi T_\phi = \lambda T_\phi$ for some $\lambda > 1$. Then for every representative $\Phi \in \mathrm{Aut}(\mathbb{F})$ of the outer automorphism class ϕ the trees ΦT_ϕ and λT_ϕ are $\mathbb{F}$–equivariantly isometric. The metric completions $\overline{\Phi T_\phi}$ and $\lambda \overline{T_\phi}$ are thus $\mathbb{F}$–equivariantly isometric as well.

Using the definition of $\overline{\Phi T_\phi}$ as an $\mathbb{F}$–tree, it follows that there exists a bijective λ–homothety $H_\Phi\colon \overline{T_\phi} \to \overline{T_\phi}$ which *represents* Φ in the sense that for every $x \in \overline{T_\phi}$ and every $w \in \mathbb{F}$ we have

$$H_\Phi(wx) = \Phi^{-1}(w) H_\Phi(x). \tag{4}$$

Moreover, there is a unique point $C(H_\Phi) \in \overline{T_\phi}$ which is fixed by H_Φ; this point is called the *center* of H_Φ.

It is known that for every representative $\Phi \in \mathrm{Aut}(\mathbb{F})$ of ϕ there exists a unique homothety H_Φ representing Φ in the above sense. Moreover, it is also known that the set of homotheties representing all representatives of ϕ in $\mathrm{Aut}(\mathbb{F})$ is exactly the set

$$\{wH_{\Phi_0} | w \in \mathbb{F}\}$$

where Φ_0 is some representatives of ϕ in $\mathrm{Aut}(\mathbb{F})$. We refer the reader to [KL3] for details.

We will need a number of known results relating homotheties H_Φ to the map $\mathcal{Q}_{T_\phi}$ which are summarized in Proposition 3.14 below. Before stating this proposition recall that there is a notion of a *forward rotationless*, or FR, element of $\mathrm{Out}(\mathbb{F})$ which allows one to disregard certain periodicity and permutational phenomena that otherwise complicate the index theory for $\mathrm{Out}(\mathbb{F})$. The notion of an FR element of $\mathrm{Out}(\mathbb{F})$ was first introduced by Feighn and Handel [FH]. We refer the reader to Definition 3.2 in [CH1] for a precise definition. For our purposes we only need to know that for every fully irreducible $\phi \in \mathrm{Out}(\mathbb{F})$ there exists $k \geq 1$ such that ϕ^k is FR [CH1, Proposition 3.3]. Note that in this case $T_\phi = T_{\phi^k}$, $L(T_\phi) = L(T_{\phi^k})$ and $\mathcal{Q}_{T_\phi} = \mathcal{Q}_{T_{\phi^k}}$. Also if $\phi \in \mathrm{Out}(\mathbb{F})$ is an FR element then ϕ^m is also FR for every $m \geq 1$.

Proposition 3.14. *Let $\phi \in \mathrm{Out}(\mathbb{F})$ be an atoroidal fully irreducible FR element.*

(1) [CH1, Proposition 3.1] *For every representative $\Phi \in \mathrm{Aut}(\mathbb{F})$ of ϕ, the left action of Φ on $\partial\mathbb{F}$ has finitely many fixed points, each of which is either a local attractor or a local repeller. Moreover, the action of Φ on $\partial\mathbb{F}$ has at least one fixed point which is a local attractor, and at least one fixed point which is a local repeller.*

(2) [CH1, Lemma 4.3] *Let $\Phi \in \mathrm{Aut}(\mathbb{F})$ be a representative of ϕ, and denote by $att(\Phi)$ the set of all fixed points of Φ in $\partial\mathbb{F}$ that are local attractors. Let H_Φ be the homothety of T_ϕ representing Φ. Then*

$$\mathcal{Q}_{T_\phi}(att(\Phi)) = C(H_\Phi) \qquad \text{and} \qquad \mathcal{Q}_{T_\phi}^{-1}(C(H_\Phi)) = att(\Phi).$$

Corollary 3.15. *Let $\phi \in \mathrm{Out}(\mathbb{F})$ be an atoroidal fully irreducible and let $\Phi \in \mathrm{Aut}(\mathbb{F})$ be a representative of ϕ. Let $p \in att(\Phi)$. Then:*

1. *The point p is proximal for $L(T_\phi)$.*
2. *If $T \in \overline{cv}$ is a free arational tree such that $L(T) \neq L(T_\phi)$ then there does not exist $p' \in \partial\mathbb{F}$ such that $(p', p) \in L(T)$.*

Proof. Proposition 3.14 implies that $Q_{T_\phi}(p) = C(H_\Phi) \in \overline{T_\phi}$. Therefore by part (1) of Proposition 3.11, p is proximal for $L(T_\phi)$, as required.

We now prove part (2) of the corollary. Let $T \in \overline{\mathrm{cv}}$ be a free arational tree such that $L(T) \neq L(T_\phi)$. Suppose that there exists $p \neq p' \in \partial\mathbb{F}$ such that $(p', p) \in L(T)$. Since p is proximal for $L(T_\phi)$, there exist X–leaf segments v_n for $L(T_\phi)$ with $|v_n| \to \infty$ as $n \to \infty$, such that each v_n occurs infinitely many times as a subword in the geodesic ray from 1 to p in $\mathrm{Cay}(\mathbb{F}, X)$. Since (p', p) is a leaf of $L(T)$, it follows that each v_n is also a leaf-segment for $L(T)$.

For each $n \geq 1$ choose a geodesic segment $\gamma_n = [u_n, w_n]$ in $\mathrm{Cay}(\mathbb{F}, X)$ with label v_n and passing through the vertex $1 \in \mathbb{F}$ such that $d_{\mathrm{Cay}(\mathbb{F},X)}(u_n, 1) \to \infty$ and $d_{\mathrm{Cay}(\mathbb{F},X)}(1, w_n) \to \infty$ as $n \to \infty$. After passing to a subsequence, we may assume that the segments γ_n converge to a bi-infinite geodesic from $u \in \partial\mathbb{F}$ to $w \in \partial\mathbb{F}$.

Since v_n is a leaf-segment for $L(T_\phi)$ there exists a sequence $(s_n, s'_n) \in L(T_\phi)$ such that the geodesic from s_n to s'_n in $\mathrm{Cay}(\mathbb{F}, X)$ passes through γ_n for every $n \geq 1$. Similarly, since v_n is a leaf-segment for $L(T)$, there exists a $(t_n, t'_n) \in L(T)$ such that the geodesic from t_n to t'_n in $\mathrm{Cay}(\mathbb{F}, X)$ passes through γ_n for every $n \geq 1$. By construction it then follows that

$$\lim_{n\to\infty} (s_n, s'_n) = \lim_{n\to\infty} (t_n, t'_n) = (u, w).$$

Since $L(T_\phi)$ and $L(T)$ are closed in $\partial^2\mathbb{F}$, it follows that $(u, w) \in L(T_\phi) \cap L(T)$. However, since T_ϕ, T are free arational trees with $L(T_\phi) \neq L(T)$, this contradicts the conclusion $L(T_\phi) \cap L(T) = \varnothing$ of Proposition 3.7. □

4. Hyperbolic extensions of free groups

For the duration of this paper, we assume that $\mathbb{F}$ is a finite-rank free group with $\mathrm{rank}(\mathbb{F}) \geq 3$. Note that if $F_2 = F(a, b)$ is free of rank two, then for every $\phi \in \mathrm{Out}(F_2)$ we have $\phi([g]) = [g^{\pm 1}]$ where $g = [a, b]$. For this reason if $1 \to F_2 \to E \to Q \to 1$ is a short exact sequence with Q and E hyperbolic, then $|Q| = [E : F_2] < \infty$. On the other hand, free groups of rank at least 3 admit many interesting word-hyperbolic extensions, as discussed in more detail below.

4.1. Subgroups of $\mathrm{Out}(\mathbb{F})$ and hyperbolic extension of free groups.

We now recall a general class of hyperbolic $\mathbb{F}$–extensions constructed in [DT1]. These hyperbolic extensions are the natural generalization of hyperbolic free-by-cyclic groups with fully irreducible monodromy. For any $\Gamma \leq \mathrm{Out}(\mathbb{F})$ there is an $\mathbb{F}$–extension E_Γ obtained from the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{F} & \xrightarrow{i} & \mathrm{Aut}(\mathbb{F}) & \xrightarrow{p} & \mathrm{Out}(\mathbb{F}) & \longrightarrow & 1 \\ & & \| & & \uparrow & & \uparrow & & \\ 1 & \longrightarrow & \mathbb{F} & \xrightarrow{i} & E_\Gamma & \xrightarrow{p} & \Gamma & \longrightarrow & 1 \end{array} \tag{5}$$

We say that $E_\Gamma := p^{-1}(\Gamma)$ is the $\mathbb{F}$–extension corresponding to Γ.

Recall that $\phi \in \mathrm{Out}(\mathbb{F})$ is called *atoroidal* if no positive power of ϕ fixes a conjugacy class of $\mathbb{F}$. A key result of Brinkmann [Bri] shows that for a cyclic subgroup $\langle\phi\rangle \leq \mathrm{Out}(\mathbb{F})$, the extension $E_{\langle\phi\rangle}$ is word-hyperbolic if and only if ϕ is atoroidal or finite order. A subgroup $\Gamma \leq \mathrm{Out}(\mathbb{F})$ is said to be *purely atoroidal* if every infinite order element of Γ is atoroidal.

The following theorem gives geometric conditions on a subgroup $\Gamma \leq \mathrm{Out}(\mathbb{F})$ that imply the corresponding extension E_Γ is hyperbolic:

Theorem 4.1 (Dowdall–Taylor [DT1])**.** *Let $\Gamma \le \mathrm{Out}(\mathbb{F})$ be finitely generated. Suppose that Γ is purely atoroidal and that for some $A \in \mathcal{F}^0$ the orbit map $\Gamma \to \mathcal{F}$ given by $g \mapsto gA$ is a quasi-isometric embedding. Then the corresponding extension E_Γ is hyperbolic.*

Recall that we have called a finitely generated subgroup $\Gamma \le \mathrm{Out}(\mathbb{F})$ *convex cocompact* if some orbit map $\Gamma \to \mathcal{F}$ is a quasi-isometric embedding. Hence, Theorem 4.1 implies that if Γ is convex cocompact, then E_Γ is hyperbolic if and only if Γ is purely atoroidal. Note that if $\phi \in \mathrm{Out}(\mathbb{F})$ is fully irreducible, then $\langle \phi \rangle$ is convex cocompact by Theorem 3.5.

Remark 4.2 (Reformulation in terms of the co-surface graph)**.** In [DT2], the authors reformulate Theorem 4.1 in terms of the co-surface graph $\mathcal{CS}$. This is the $\mathrm{Out}(\mathbb{F})$–graph defined as follows: vertices are conjugacy classes of primitive elements of $\mathbb{F}$ and two conjugacy classes α and β are joined by an edge whenever there is a once punctured surface S whose fundamental group can be identified with $\mathbb{F}$ in such a way that α and β both represent simple closed curves on S. We note that closely related graphs appear in [KL1, MR1, Man]; see [DT2] for a discussion and further references. In [DT1, Theorem 9.2], it is shown that if $\Gamma \le \mathrm{Out}(\mathbb{F})$ admits a quasi-isometric orbit map into $\mathcal{CS}$, then Γ is purely atoroidal and convex cocompact, and hence the corresponding extension E_Γ is hyperbolic. In [DT2], the converse is proven: A finitely generated subgroup $\Gamma \le \mathrm{Out}(\mathbb{F})$ admits a quasi-isometric orbit map into the co-surface graph if and only if Γ is purely atoroidal and convex cocompact. The authors in [DT2] use this characterization to further study the geometry of the hyperbolic extension E_Γ.

4.2. **Laminations for hyperbolic extensions.** Fix $\Gamma \le \mathrm{Out}(\mathbb{F})$ finitely generated such that the corresponding extension

$$1 \longrightarrow \mathbb{F} \longrightarrow E_\Gamma \longrightarrow \Gamma \longrightarrow 1$$

is an exact sequence of hyperbolic groups.

Definition 4.3 (Mitra's laminations)**.** Let $z \in \partial\Gamma$. Let ρ be a geodesic ray in Γ from 1 to z, with the vertex sequence $g_1, g_2, g_3, \dots, g_n, \dots$ in Γ.

For $1 \ne h \in \mathbb{F}$ let w_n be a cyclically reduced word over $X^{\pm 1}$ representing the conjugacy class $[g_n(h)]$ in $\mathbb{F}$. Let $R_{z,h}$ be the set of all pairs $(u, u') \in \mathbb{F} \times \mathbb{F}$ such that the freely reduced form v of $u^{-1}u'$ occurs a subword in a cyclic permutation of w_n or of w_n^{-1} for some $n \ge 1$. Put

$$\Lambda_{z,h} = \overline{R_{z,h}} \cap \partial^2\mathbb{F}$$

where $\overline{R_{z,h}}$ is the closure of $R_{z,h}$ in $(\mathbb{F} \cup \partial\mathbb{F}) \times (\mathbb{F} \cup \partial\mathbb{F})$. Thus $\Lambda_{z,h}$ consists of all $(p_1, p_2) \in \partial^2\mathbb{F}$ such that there exists a sequence $(u_i, u_i') \in \mathbb{F} \times \mathbb{F}$ converging to (p_1, p_2) in $(\mathbb{F} \cup \partial\mathbb{F}) \times (\mathbb{F} \cup \partial\mathbb{F})$ as $i \to \infty$ and such that for every $i \ge 1$ the freely reduced form v_i of $(u_i)^{-1}u_i'$ occurs as a subword in a cyclic permutation of some w_{n_i} or of $w_{n_i}^{-1}$ (which, since $p_1 \ne p_2$, automatically implies that $n_i \to \infty$ as $i \to \infty$). Put

$$\Lambda_z := \bigcup_{h \in \mathbb{F} \setminus \{1\}} \Lambda_{z,h}.$$

Finally, define the *ending lamination* of the extension to be

$$\Lambda := \bigcup_{z \in \partial\Gamma} \Lambda_z.$$

Remark 4.4. Thus $\Lambda_{z,h}$ consists of all $(p, q) \in \partial^2\mathbb{F}$ such that for every subword v of the bi-infinite geodesic from p to q in $\mathrm{Cay}(\mathbb{F}, X)$ there exists $m \ge 1$ such that v is a subword of a cyclic permutation of w_m or w_m^{-1}.

Mitra [Mit1, Lemma 3.3] shows that the definition of Λ_z does not depend on the choice of a geodesic ray $(g_n)_n$ from 1 to z in the Cayley graph of Γ. Moreover, the proof of [Mit1, Lemma 3.3] implies that instead of a geodesic ray one can also use any quasigeodesic sequence from 1 to z in Γ. [Mit1, Remark on p. 399] also shows that for every $z \in \partial\Gamma$ there exists a finite subset $R \subseteq \mathbb{F} \setminus \{1\}$

such that $\Lambda_z = \cup_{h\in R}\Lambda_{z,h}$. Since every $\Lambda_{z,h}$ is an algebraic lamination on $\mathbb{F}$, it follows that Λ_z is also an algebraic lamination on $\mathbb{F}$.

The results of Mitra [Mit1] imply that $\Lambda_{z,h}$ does not depend on the choice of a free basis X of $\mathbb{F}$. Therefore Λ_z and Λ are independent of X as well. In Lemma 4.5 below we give an equivalent definition of $\Lambda_{z,h}$ which does not involve the choice of X; this gives another proof that $\Lambda_{z,h}$ is independent of X.

Mitra [Mit1] in fact defines $\Lambda_{z,h}$, Λ_z and Λ in the context of an arbitrary short exact sequence of word-hyperbolic groups. As it suffices for our purposes, here we have only presented the definitions in the somewhat more transparent setting free group extensions.

Recall that for a collection Ω of conjugacy classes of $\mathbb{F}$, we define $L(\Omega)$ to be the smallest algebraic lamination containing $L(g)$ for each $g \in \Omega$.

Lemma 4.5. *For $z \in \partial\Gamma$, let $(g_i)_{i\ge 0}$ be a geodesic ray in Γ such that $\lim_{i\to\infty} g_i = z \in \partial\Gamma$. Then for any $h \in \mathbb{F}\setminus\{1\}$ we have*

$$\Lambda_{z,h} = \bigcap_{k\ge 0} L\big(\{g_i(h) : i \ge k\}\big).$$

Proof. Let w_n be the cyclically reduced form over $X^{\pm 1}$ of $g_n(h)$. Recall that $\Lambda_{z,h}$ consists of all $(p,q) \in \partial^2\mathbb{F}$ such that for every finite subword v of the bi-infinite geodesic in $\mathrm{Cay}(\mathbb{F},X)$ from p to q there exists $n \ge 1$ such that v is a subword of a cyclic permutation of w_n or of w_n^{-1}.

Put $L := \bigcap_{k\ge 0} L(\{g_i(h) : i \ge k\})$. Then L consists of all $(p,q) \in \partial^2\mathbb{F}$ such that for every finite subword v of the bi-infinite geodesic in $\mathrm{Cay}(\mathbb{F},X)$ from p to q and every $M \ge 1$ there exist $n \ge M$ and $m \in \mathbb{Z}\setminus\{0\}$ such that v is a subword of a cyclic permutation of w_n^m. Hence $\Lambda_{z,h} \subseteq L$.

Let $(p,q) \in L$ be arbitrary. Let v be a finite subword of of the bi-infinite geodesic in $\mathrm{Cay}(\mathbb{F},X)$ from p to q. We will use the following claim to complete the proof:

Claim. *The cyclically reduced length $\|w_n\|$ of w_n tends to ∞ as $n \to \infty$.*

Assuming the claim, choose $M \ge 1$ such that for all $n \ge M$ we have $\|w_n\| \ge |v|$. Since $(p,q) \in L$, there exist $n \ge M$ and $m \in \mathbb{Z}\setminus\{0\}$ such that v is a subword of a cyclic permutation of w_n^m. The fact that $\|w_n\| \ge |v|$ implies that v is a subword of a cyclic permutation of w_n or of w_n^{-1}. Therefore $(p,q) \in \Lambda_{z,h}$. Hence $L \subseteq \Lambda_{z,h}$ and so $L = \Lambda_{z,h}$, as required.

We now prove the claim. Note that since E_Γ is hyperbolic, each infinite order element of Γ is atoroidal. Hence, if we denote by Γ_α the subgroup of Γ consisting of those elements which fix the conjugacy class α, then Γ_α is a torsion subgroup of $\mathrm{Out}(\mathbb{F})$ and hence by [DT1, Lemma 2.13] has $|\Gamma_\alpha| \le e$ for some $e \ge 0$ depending only on $\mathrm{rank}(\mathbb{F})$. If the claim is false, then there is a $D \ge 0$ and a infinite subsequence such that $\|g_{n_i}(h)\| \le D$ for all $i \ge 0$. Let C denote the finite number (depending on D and X) of conjugacy classes of $\mathbb{F}$ whose cyclically reduced length is at most D. Hence if $k \ge C(e+2)$, we may find at least $e+2$ distinct elements in the list $g_{n_1}(h), \dots, g_{n_k}(h)$ that all belong to the same conjugacy class α. This produces $e+1$ distinct elements of Γ which fix the conjugacy class α. This contradicts our choice of e and completes the proof of the claim. ☐

The main result of Mitra in [Mit1] is:

Theorem 4.6. [Mit1, Theorem 4.11] *Suppose that $1 \to H \to G \to \Gamma \to 1$ is an exact sequence of hyperbolic groups with Cannon–Thurston map $\partial i : \partial H \to \partial G$. Then for distinct points $p, q \in \partial H$, $\partial i(p) = \partial i(q)$ if and only if $(p,q) \in \Lambda$ if and only if $(p,q) \in \Lambda_z$ for some $z \in \partial\Gamma$.*

5. Dual laminations at the boundary of Γ

Convention 5.1. For the remainder of this paper, we fix a free group $\mathbb{F}$ of finite rank at least 3 and a finitely generated, purely atoroidal, convex cocompact subgroup $\Gamma \le \mathrm{Out}(\mathbb{F})$. Thus we may choose a cyclic free factor $x \in \mathcal{F}^0$ so that the orbit map $\Gamma \to \mathcal{F}$ given by $g \mapsto gx$ is a quasi-isometric embedding. This orbit map then induces a Γ–equivariant topological embedding $\kappa\colon \partial\Gamma \to \partial\mathcal{F}$ and we identify $\partial\Gamma$ with its image in $\partial\mathcal{F}$. Hence, each point $z \in \partial\Gamma$ correspond to equivalence class T_z of arational trees, each of which has a well-defined dual lamination $L(T_z)$ (Definition 3.2). Furthermore, Theorem 4.1 shows that the corresponding extension E_Γ is a hyperbolic group. Thus the short exact sequence

$$1 \longrightarrow \mathbb{F} \longrightarrow E_\Gamma \longrightarrow \Gamma \longrightarrow 1$$

of hyperbolic groups admits a surjective Cannon–Thurston map $\partial\iota \colon \partial\mathbb{F} \to \partial E_\Gamma$ by Proposition-Definition 2.1, and for every $z \in \partial\Gamma$ there is a corresponding ending lamination Λ_z as defined by Mitra in Definition 4.3

The main result of this section characterizes the laminations $\{\Lambda_z \colon z \in \partial\Gamma\}$ appearing in Theorem 4.6 for the extension $1 \to \mathbb{F} \to E_\Gamma \to \Gamma \to 1$. Recall that we have denoted by $\partial\pi : \mathcal{AT} \to \partial\mathcal{F}$ the map which associates to each arational tree of $\overline{\text{cv}}$ the corresponding point in the boundary of the factor complex (see Theorem 3.6).

Theorem 5.2. *For each $z \in \partial\Gamma$, there is $T_z \in \overline{cv}$ which is free and arational such that $z \mapsto \partial\pi(T_z)$ under $\partial\Gamma \to \partial\mathcal{F}$ with the property that*

$$\Lambda_z = L(T_z).$$

Corollary 5.3. *Let $\Gamma \leq \mathrm{Out}(\mathbb{F})$ be convex cocompact and purely atoroidal. Then the Cannon–Thurston map $\partial\iota \colon \partial\mathbb{F} \to \partial E_\Gamma$ identifies points $a, b \in \partial\mathbb{F}$ if and only if there exists $z \in \partial\Gamma$ such that $(a, b) \in L(T_z)$. That is, $\partial\iota$ factors through the quotient of $\partial\mathbb{F}$ by the equivalence relation*

$$a \sim b \iff (a, b) \in L(T_z) \text{ for some } z \in \partial\Gamma$$

and descends to an E_Γ–equivariant homeomorphism $\partial\mathbb{F}/\sim \to \partial E_\Gamma$.

Proof. The specified equivalence relation is by definition given by the subset $\bigcup_{z\in\partial\Gamma} L(T_z) = \Lambda$ of $\partial^2\mathbb{F}$, where the last equality holds by Theorem 5.2. Theorem 4.6 asserts that $\Lambda = \{(p, q) \in \partial\mathbb{F} \times \partial\mathbb{F} : \partial\iota(p) = \partial\iota(q)\}$. Since the Cannon–Thurston map $\partial\iota : \partial\mathbb{F} \to \partial E_\Gamma$ is continuous, it follows that Λ is a closed subset of $\partial\mathbb{F} \times \partial\mathbb{F}$. Therefore $\partial\mathbb{F}/\sim$, equipped with the quotient topology, is a compact Hausdorff topological space. Moreover, the continuity and surjectivity of $\partial\iota \colon \partial\mathbb{F} \to \partial E_\Gamma$ now imply that $\partial\iota$ quotients through to a continuous bijective map $J \colon \partial\mathbb{F}/\sim \to \partial E_\Gamma$, which is, by construction, E_Γ–equivariant. The fact that both $\partial\mathbb{F}/\sim$ and ∂E_Γ are compact Hausdorff topological spaces implies that J is a homeomorphism, as required. ☐

Recall that by a general result of [CHL2], for every $z \in \partial\Gamma$ the map $\mathcal{Q}_{T_z} \colon \partial\mathbb{F} \to \hat{T}_z$ quotients through to a $\mathbb{F}$–equivariant homeomorphism $\partial\mathbb{F}/L(T_z) \to \hat{T}_z$, where $\partial\mathbb{F}/L(T_z)$ is given the quotient topology and where $\hat{T}_z$ is given the "observer's topology". Now a similar argument to the proof of Corollary 5.3 implies the following statement (we leave the details to the reader):

Corollary 5.4. *For each $z \in \partial\Gamma$ the Cannon–Thurston map $\partial\iota \colon \partial\mathbb{F} \to E_\Gamma$ factors through $\mathcal{Q}_z \colon \partial\mathbb{F} \to \hat{T}_z$ and induces a continuous, surjective $\mathbb{F}$–equivariant map $\hat{T}_z \to \partial E_\Gamma$ (where $\hat{T}_z$ is equipped with the observer's topology).*

We now start working towards the proof of Theorem 5.2. For our next lemma we assume that the reader has some familiarity with folding paths in $\mathcal{X}$; for example [BF2, Section 2, 4]. This material is also summarized in [DT1, Section 2.7] and the reader may find helpful the discussion appearing before Lemma 6.9 of [DT1]. For a folding path G_t, we say that a conjugacy class α is *mostly legal* at time t_0 if its *legal length* $\mathrm{leg}(\alpha|G_{t_0})$ is at least half of its total length $\ell(\alpha|G_{t_0})$. Of course, if α is mostly legal at time t_0, then it is mostly legal for all $t \geq t_0$. Also, for any path $q \colon \mathbf{I} \to \mathcal{X}$, we say that q has the (λ, N_0)–flaring property for constants $\lambda, N_0 \geq 1$ if for any $t \in \mathbf{I}$ and any $\alpha \in \mathbb{F} \setminus \{1\}$

$$\lambda \cdot \ell(\alpha|q(t)) \leq \max\big\{\ell(\alpha|q(t - N_0)), \ell(\alpha|q(t + N_0))\big\}.$$

Fix a rose $R \in \mathcal{X}$ with a petal labeled by our fixed $x \in \mathcal{F}^0$. Observe that x is contained in the projection $\pi(R)$ of R to the factor complex $\mathcal{F}$.

Lemma 5.5 (Everyone's eventually mostly legal)**.** *For $\Gamma \leq \mathrm{Out}(\mathbb{F})$ as in Convention 5.1 and for any $K \geq 0$ and $\lambda > 1$, there exist $N_0, c \geq 1$ satisfying the following: Suppose that $\gamma\colon \mathbf{I} \to \mathcal{X}$ is a unit speed folding path contained in a symmetric K–neighborhood of $\Gamma \cdot R \subset \mathcal{X}$. Then $\gamma\colon \mathbf{I} \to \mathcal{X}$ has the (λ, N_0)–flaring property. Moreover, for any conjugacy class α whose length along γ is minimized at $t_\alpha \in \mathbf{I}$, we have for all $t \geq t_\alpha$*

$$\frac{1}{c} \cdot e^{(t-t_\alpha)} \ell(\alpha|\gamma(t_\alpha)) \leq \ell(\alpha|\gamma(t)) \leq e^{(t-t_\alpha)} \ell(\alpha|\gamma(t_\alpha)).$$

Proof. That $\gamma\colon \mathbf{I} \to \mathcal{X}$ has the (λ, N_0)–flaring property is exactly the conclusion of Proposition 6.11 of [DT1] (note that the needed "A_0–QCX" hypothesis follows from [DT1, Corollary 6.3]). Also, the upper bound in the statement of the lemma follows immediately from the definition of a unit speed folding path (see [BF2, Section 4]), so we focus on the lower bound.

For the conjugacy class α, let s_α be the infimum of times for which α is mostly legal (if such a time does not exist, set s_α to be the right endpoint of $\mathbf{I}$). We show that $s_\alpha - t_\alpha \leq C$, for some constant C not depending on α. Then [DT1, Lemma 6.10] (which is an application of [BF2, Corollary 4.8]) implies that for $t \geq t_\alpha$

$$\begin{aligned} \ell(\alpha|\gamma(t)) &\geq \frac{1}{3} e^{t-s_\alpha} \mathrm{leg}(\alpha|\gamma(s_\alpha)) \\ &\geq \frac{1}{6} e^{t-s_\alpha} \ell(\alpha|\gamma(s_\alpha)) \\ &= \frac{1}{6} e^{t-t_\alpha} e^{-(s_\alpha - t_\alpha)} \ell(\alpha|\gamma(s_\alpha)) \\ &\geq \frac{1}{6e^C} \cdot e^{t-t_\alpha} \ell(\alpha|\gamma(t_\alpha)), \end{aligned}$$

as needed. Hence, it suffice to prove the uniform bound $s_\alpha - t_\alpha \leq C$ over all nontrivial conjugacy classes α. This will follow from applying the flaring property of the folding path γ; the idea is that if α is not mostly legal at some time t_0 then either the length of α decreases at some definite rate at t_0 (which is impossible if $t_0 = t_\alpha$), or after a bounded amount of time α becomes mostly legal. The details are slightly technical and our argument relies on the proof of Proposition 6.11 of [DT1].

Since the image of γ is contained in the K–neighborhood (with respect to $d_{\mathcal{X}}^{\mathrm{sym}}$) of $\Gamma \cdot R$, there is an $\epsilon > 0$ depending only on $R \in \mathcal{X}$ and $K \geq 1$ such that $\gamma(\mathbf{I}) \subset \mathcal{X}_{\geq \epsilon}$, the ϵ–thick part of $\mathcal{X}$. The flaring property then implies that there is a $M \geq 1$ depending only on λ and ϵ such that

$$12 \leq \ell(\alpha|\gamma(t_0 - N_0)) \leq \frac{1}{\lambda} \ell(\alpha|\gamma(t_0)), \tag{6}$$

for $t_0 = t_\alpha + M$. Hence, it suffices to bound the difference $s_\alpha - t_0$. According to the proof of Proposition 6.11 of [DT1] either (1) $\mathrm{ilg}(\alpha|\gamma(t_0)) \geq \frac{\ell(\alpha|\gamma(t_0))}{2}$, or (2) $\mathrm{ilg}(\alpha|\gamma(t_0)) < \frac{\ell(\alpha|\gamma(t_0))}{2}$ and $\mathrm{leg}(\alpha|\gamma(t_0)) > 0$. Here, $\mathrm{ilg}(\alpha|\gamma(t_0))$ is the illegal length of α as defined in Section 6 of [DT1]. (We note that the third case of [DT1, Proposition 6.11] does not arise since in that case $\ell(\alpha|\gamma(t_0)) < 6$.)

In case (1), Proposition 6.11 shows that $\ell(\alpha|\gamma(t_0 - N_0)) \geq \lambda \cdot \ell(\alpha|\gamma(t_0))$, which directly contradicts (6). Hence, we conclude that we are in the situation of case (2) of [DT1, Proposition 6.11], where it is shown that the legal length constitutes a definite fraction of the total length of α in $\gamma(t_0)$. In fact, there it is shown that

$$\mathrm{leg}(\alpha|\gamma(t_0)) \geq \frac{\ell(\alpha|\gamma(t_0)) - 6}{2(1 + \breve{m})} \geq \frac{\ell(\alpha|\gamma(t_0))}{4(1 + \breve{m})},$$

where $\breve{m}$ is a constant depending only on the rank of $\mathbb{F}$. From this it follows easily that $s_\alpha - t_0$ is uniformly bounded (e.g., [DT1, Lemmas 6.9–6.10] show that illegal length decays at a definite rate whereas legal length grows at a definite rate). This completes the proof of the lemma. □

The companion to Lemma 5.5 is the following proposition, which states that we can extract folding rays in $\mathcal{X}$ which stay uniformly close to the orbit of Γ, have the required flaring property, and limit to free, arational trees in $\partial\mathcal{X}$. Most of this follows from the main technical work in [DT1] on stable quasigeodesics in $\mathcal{X}$. Recall we have fixed $R \in \mathcal{X}$ with a petal labeled by x.

Proposition 5.6 (Folding rays to infinity)**.** *For any $k, \lambda \geq 1$ there are $M, K \geq 0$ such that if $(g_i)_{i\geq 0}$ is a k–quasigeodesic ray in Γ, then there is an infinite length folding ray $\gamma\colon \mathbf{I} \to \mathcal{X}$ parameterized at unit speed with the following properties:*

1. *The sets $\gamma(\mathbf{I})$ and $\{g_i R : i \geq 0\}$ have symmetric Hausdorff distance at most K.*
2. *The rescaled folding path $G_t = e^{-t}\cdot\gamma(t) \in \mathrm{cv}$ converges to the arational tree $T \in \partial\,\mathrm{cv}$ with the property that $\lim_{i\to\infty} g_i x = \partial\pi(T)$ in $\mathcal{F} \cup \partial\mathcal{F}$, where $\partial\pi(T)$ is the projection of the projective class of T to the boundary of $\mathcal{F}$. Moreover, the action $\mathbb{F} \curvearrowright T$ is free.*
3. *The folding path γ has the (λ, M) flaring property.*

Proof. By Theorem 5.5 of [DT1], the orbit $\Gamma \cdot R$ is quasiconvex; hence, there is a $K \geq 0$ depending only on $k \geq 0$ (and the quasi-isometry constants of the orbit map $\Gamma \to \mathcal{F}$) such that any geodesic of $\mathcal{X}$ joining points of $(g_i R)_{i\geq 0}$ is contained in the symmetric K–neighborhood of the quasigeodesic $(g_i R)_{i\geq 0}$ (note that Γ is word-hyperbolic and moreover qi-embedded into $\mathcal{X}$ by [DT1, Lemma 6.4]). Let γ_i be a standard geodesic of $\mathcal{X}$ joining $g_0 R$ to $g_i R$. Since this collection of geodesics begins at $g_0 R$ and remains in a symmetric K–neighborhood of $(g_i R)_{i\geq 0}$, the Arzela–Ascoli theorem implies that (after passing to a subsequence) the γ_i converge uniformly on compact sets to a geodesic ray $\gamma\colon \mathbf{I} \to \mathcal{X}$, which is also contained in the symmetric K–neighborhood of $(g_i R)_{i\geq 0}$. Hence, the geodesic γ is contained in $\mathcal{X}_{\geq\epsilon}$ for ϵ depending only on K. As in the proof of Lemma 6.11 of [BR2], we see that except for some initial portion of γ of uniformly bounded size, γ is a folding path. Hence, up to increasing K by a bounded amount, this completes the proof of item (1).

To prove (2), let ξ denote the limit of $(g_i x)_{i\geq 0}$ in $\partial\mathcal{F}$. Note that the rescaled folding path $G_t = e^{-t}\cdot\gamma(t)$ is isometric on edges and hence converges to a tree $T \in \overline{\mathrm{cv}}$ [HM]. Hence $\gamma(t)$ converges to the projective class of T in $\overline{\mathcal{X}}$ as $t \to \infty$ and by item (1)

$$\lim_{t\to\infty} \pi(\gamma(t)) = \lim_{i\to\infty} \pi(g_i R) = \lim_{i\to\infty} g_i x = \xi,$$

in $\mathcal{F} \cup \partial\mathcal{F}$. Hence, the tree T is arational and $\partial\pi(T) = \xi \in \partial\mathcal{F}$ by Theorem 3.6. To compete the proof of (2), it only remains to show that the tree T has a free $\mathbb{F}$–action. This will follow using item (3), which we note follows immediately from Proposition 6.11 of [DT1].

To see that $\mathbb{F} \curvearrowright T$ is free, it suffices to show that $\ell_T(\alpha) > 0$ for each $\alpha \in \mathbb{F} \setminus \{1\}$. Since $\lim_{t\to\infty} G_t = T$, we see using (3) and Lemma 5.5 that

$$\begin{aligned}
\ell_T(\alpha) &= \lim_{t\to\infty} \ell(\alpha|G_t) \\
&= \lim_{t\to\infty} e^{-t}\cdot\ell(\alpha|\gamma(t)) \\
&\geq \lim_{t\to\infty} e^{-t}\cdot\frac{e^{(t-t_\alpha)}}{c}\ell(\alpha|\gamma(t_\alpha)) \\
&= \frac{1}{ce^{t_\alpha}}\cdot\ell(\alpha|\gamma(t_\alpha)) \\
&> 0.
\end{aligned}$$

This completes the proof. □

Proposition 5.6, together with the fact that every fully irreducible element of $\mathrm{Out}(\mathbb{F})$ acts on $\mathcal{F}$ as a loxodromic isometry, immediately implies:

Corollary 5.7. *Let $\phi \in \Gamma$ be an element of infinite order. Then for the point $z = \phi^\infty \in \partial\Gamma$ we have $T_z = T_\phi$. That is, z is mapped under the map $\partial\Gamma \to \partial\mathcal{F}$ to the $\approx$–equivalence class $[T_\phi] \in \partial\mathcal{F}$, where T_ϕ is the stable tree of ϕ.*

Most of the work of Theorem 5.2 is done with the following proposition.

Proposition 5.8. *Let $(g_i)_{i\geq 0}$ be a quasigeodesic sequence in Γ converging to $z \in \partial\Gamma$. Then there is a $T_z \in \overline{cv}$ such that $z \mapsto \partial\pi_1(T_z)$ under $\partial\Gamma \to \partial\mathbb{F}$ with the property that for every $1 \neq h \in \mathbb{F}$ we have*

$$\lim_{i\to\infty} \ell_{T_z}(g_i h) = 0.$$

Proof. Apply Proposition 5.6 with $\lambda = 2$ to obtain the unit speed folding ray $\gamma\colon \mathbf{I} \to \mathfrak{X}$ and denote by $G_t = e^{-t}\cdot\gamma(t)$ the rescaled folding path for which the associated folding maps are isometric on edges. Let T_z be the limit of G_t in $\overline{\text{cv}}$. By Proposition 5.6, the image of z under the extension of the orbit map $\partial\Gamma \to \partial\mathcal{F}$ is $\partial\pi(T_z)$.

As in Lemma 5.5, let t_{g_ih} denote a time for which g_ih has its length minimized along $\gamma(t)$. Also, for each $i \geq 0$ let t_i denote a time for which the symmetric distance between $\gamma(t)$ and g_iR is less than K. (Such a time exists by Proposition 5.6.) Hence, by definition of the symmetric distance on $\mathfrak{X}$ we have

$$e^{-K} \leq \frac{\ell(\alpha|\gamma(t_i))}{\ell(\alpha|g_iR)} \leq e^K, \tag{7}$$

for each conjugacy class α. We will need the following claim:

Claim. *There is a constant $B \geq 0$ which is independent of $i \geq 0$ so that*

$$|t_i - t_{g_ih}| \leq B.$$

Proof of claim. We will show that B can be taken to be

$$\max\left\{2M + M\log_2\frac{e^M e^K \ell(h|R)}{\epsilon},\ \log\left(\frac{c\cdot\ell(h|R)}{\epsilon}\right) + K\right\},$$

where the constants M, K, c are as in Proposition 5.6. To see this, first suppose that $t_i < t_{g_ih}$. Let $D = \left\lfloor \frac{t_{g_ih} - t_i}{M}\right\rfloor$ so that $DM \leq t_{g_ih} - t_i \leq DM + M$ and consequently

$$\ell(g_ih|\gamma(t_{g_ih} - DM)) \leq e^M \ell(g_ih|\gamma(t_i)) \tag{8}$$

since γ is a directed geodesic. As in Proposition 5.6 let ϵ be the length of the shortest loop appearing along the folding path γ. Then by definition of t_{g_ih} we have

$$\ell(g_ih|\gamma(t_{g_ih} - M)) \geq \ell(g_ih|\gamma(t_{g_ih})) \geq \epsilon.$$

Applying the $(2, M)$–flaring condition inductively at times $t_{g_ih} - M, t_{g_ih} - 2M, \dots, t_{g_ih} - DM$, we find that

$$\ell(g_ih|\gamma(t_{g_ih} - DM)) \geq 2^{D-1}\ell(g_ih|\gamma(t_{g_ih} - M)) \geq 2^{D-1}\epsilon.$$

Combining with Equation (8) and rearranging gives

$$\frac{t_{g_ih} - t_i}{M} - 2 \leq D - 1 \leq \log_2\frac{e^M\ell(g_ih|\gamma(t_i))}{\epsilon}.$$

Applying Equation (7) and isolating $t_{g_ih} - t_i$ now gives the desired bound

$$t_{g_ih} - t_i \leq 2M + M\log_2\frac{e^M e^K \ell(g_ih|g_iR))}{\epsilon} = 2M + M\log_2\frac{e^M e^K\ell(h|R)}{\epsilon}.$$

Now suppose that $t_{g_ih} \leq t_i$. Applying Lemma 5.5 to the conjugacy class $\alpha = g_ih$ and using Equation (7) then yields

$$\begin{aligned}
e^K &\geq \frac{\ell(g_ih|\gamma(t_i))}{\ell(g_ih|g_iR)} \\
&\geq \frac{\ell(g_ih|\gamma(t_{g_ih}))}{c\cdot\ell(h|R)} e^{(t_i - t_{g_ih})} \\
&\geq \frac{\epsilon}{c\cdot\ell(h|R)} e^{(t_i - t_{g_ih})}.
\end{aligned}$$

One final rearrangement then gives the claimed bound

$$t_i - t_{g_i h} \le K + \log \frac{c \cdot \ell(h|R)}{\epsilon}$$ □

We next observe that $t_i \to \infty$ as $i \to \infty$. To see this, recall that the orbit map $g \mapsto gR$ gives a quasi-isometric embedding $\Gamma \to \mathcal{X}$ (see, e.g., [DT1, Lemma 6.4]). Since $(g_i)_{i\ge 0}$ is a geodesic in Γ, this implies $d_{\mathcal{X}}(g_0R, g_iR) \to \infty$ as $i \to \infty$. Therefore

$$t_i = d_{\mathcal{X}}(\gamma(0), \gamma(t_i)) \ge d_{\mathcal{X}}(g_0R, g_iR) - 2K \to \infty \tag{9}$$

as $i \to \infty$, as claimed.

Finally, we can now compute

$$\begin{aligned}
\lim_{i\to\infty} \ell_T(g_ih) &= \lim_{i\to\infty}\lim_{t\to\infty} \ell(g_ih|G_t) \\
&= \lim_{i\to\infty}\lim_{t\to\infty} e^{-t}\cdot \ell(g_ih|\gamma(t)) \\
&\le \lim_{i\to\infty}\lim_{t\to\infty} e^{-t}(e^{(t-t_{g_ih})}\ell(g_ih|\gamma(t_{g_ih}))) && (\textit{Lemma } 5.5) \\
&= \lim_{i\to\infty} e^{-t_{g_ih}}\cdot \ell(g_ih|\gamma(t_{g_ih})) \\
&\le \lim_{i\to\infty} e^{2B}e^{-t_i}\cdot \ell(g_ih|\gamma(t_i)) && (\textit{Claim } 5) \\
&= \lim_{i\to\infty} e^{2B+K}e^{-t_i}\cdot \ell(g_ih|g_iR) && (\textit{Equation } (7)) \\
&= e^{2B+K}\ell(h|R)\cdot \lim_{i\to\infty} e^{-t_i} \\
&= 0 && (\textit{Equation } (9)).
\end{aligned}$$

This completes the proof of the proposition. □

Remark 5.9 (Unique ergodicity of T_z)**.** Although we will not need this fact, we note that Namazi–Pettet–Reynolds have recently shown that under the assumption that the orbit map $\Gamma \to \mathcal{F}$ is a quasi-isometric embedding, each equivalence class of trees appearing the image of $\partial\Gamma$ in $\partial\mathcal{F}$ is uniquely ergodic [NPR]. (See [NPR] and the references therein for a discussion of the various notions of unique ergodicity.) In particular, in the statement of Proposition 5.8 (and therefore Theorem 5.2), the tree T_z is unique up to rescaling.

We can now conclude the proof of Theorem 5.2.

Proof of Theorem 5.2. Choose a free basis X of $\mathbb{F}$. Let $z \in \partial\Gamma$ and let $(g_n)_{n=1}^{\infty}$ be a geodesic ray from 1 to z in the Cayley graph of Γ.

Let $(p,q) \in \Lambda_z$ be an arbitrary leaf of Λ_z. Then there is $1 \ne h \in \mathbb{F}$ such that $(p,q) \in \Lambda_{z,h}$. For every $n \ge 1$ let w_n be the cyclically reduced form of $g_n(h)$ over $X^{\pm 1}$. Let γ be the bi-infinite geodesic from p to q in $\mathrm{Cay}(\mathbb{F}, X)$. Let v be the label of some finite subsegment of γ. By Remark 4.4, there exists an infinite sequence $n_i \to \infty$ such that for all $i \ge 1$ v is a subword of a cyclic permutation of $w_{n_i}^{\pm 1}$. By Proposition 5.8, we see that $\lim_{i\to\infty} \ell_{T_z}(w_{n_i}) = 0$. Thus for every $\epsilon > 0$ there exists a cyclically reduced word w over $X^{\pm 1}$ with $\ell_{T_z}(w) \le \epsilon$ such that v is a subword of w. Since v was the label of an arbitrary subsegment of the geodesic from p to q, by Remark 3.3 it follows that $(p,q) \in L(T_z)$. As $(p,q) \in \Lambda_z$ was arbitrary, we conclude that $\Lambda_z \subset L(T_z)$.

Since the orbit map $\Gamma \to \mathcal{F}$ is a quasi-isometric embedding, this orbit map extends to a Γ–equivariant injective continuous map $\partial\Gamma \to \partial\mathcal{F}$. Thus for any distinct $z_1, z_2 \in \partial\Gamma$ we have $T_{z_1} \not\approx T_{z_2}$ and therefore, by Proposition 3.7, there do not exist $p, q, q' \in \partial\mathbb{F}$ such that $(p,q) \in L(T_{z_1})$ and $(p,q') \in L(T_{z_2})$. Since $\Lambda = \cup_{z\in\partial\Gamma}\Lambda_z$ is diagonally closed by Theorem 4.6, it now follows that for every $z \in \partial\Gamma$ the lamination Λ_z is diagonally closed.

Let $z \in \partial\Gamma$ be arbitrary. Proposition 3.8 implies that $L(T_z)$ is diagonally closed. Since T_z is free and arational, [CHR, Theorem A] (see also [BR2, Proposition 4.2]) implies that $L(T_z)$ possesses a unique minimal sublimation and that $L(T_z)$ is obtained from this minimal sublamination by adding diagonal leaves. Therefore the only diagonally closed sublamination of $L(T_z)$ is $L(T_z)$ itself. We have already established that $\Lambda_z \subseteq L(T_z)$. Since Λ_z is an algebraic lamination on $\mathbb{F}$ (see Remark 4.4) and since Λ_z is diagonally closed, it follows that $\Lambda_z = L(T_z)$, as required. ☐

6. Fibers of the Cannon–Thurston map

Recall (c.f. Convention 5.1) that we have fixed a convex cocompact subgroup $\Gamma \leq \mathrm{Out}(\mathbb{F})$ for which the extension group E_Γ is hyperbolic. The short exact sequence $1 \to \mathbb{F} \to E_\Gamma \to \Gamma \to 1$ thus gives rise to a surjective Cannon–Thurston map denoted $\partial\iota\colon \partial\mathbb{F} \to \partial E_\Gamma$. We write $\deg(y)$ for the cardinality $\#\big((\partial\iota)^{-1}(y)\big)$ of the fiber over $y \in \partial E_\Gamma$ and call this the *degree* of y. In this section we use Theorem 5.2 to describe the fibers of the Cannon–Thurston map. The key technical observation is the following.

Lemma 6.1. *Suppose $y \in \partial E_\Gamma$ has $\deg(y) \geq 2$. Then there is a unique point $z \in \partial\Gamma$ and a point $c \in T_z$ so that $(\partial\iota)^{-1}(y) = \mathcal{Q}_{T_z}^{-1}(c)$. Moreover, for any $p \in (\partial\iota)^{-1}(y)$ we have that p is proximal for $L(T_z)$ and that*

$$(\partial\iota)^{-1}(y) = \mathcal{Q}_{T_z}^{-1}(c) = \{p\} \cup \{q \in \partial\mathbb{F} \mid (p,q) \in L(T_z)\}.$$

Proof. Recall that since the orbit map $\Gamma \to \mathcal{F}$ is a quasi-isometric embedding and since Γ and $\mathcal{F}$ are Gromov-hyperbolic, we have a Γ–equivariant topological embedding $\kappa\colon \partial\Gamma \to \partial F$. Thus to every $z \in \partial\Gamma$ we have an associated point $\kappa(z) \in \partial\mathcal{F}$ which is represented by an equivalence class of an arational tree $T_z \in \overline{\mathrm{cv}}$. Moreover, by Theorem 5.2, for every $z \in \partial\Gamma$ the action of $\mathbb{F}$ on T_z is free and $\Lambda_z = L(T_z)$. Note that since κ is injective, Proposition 3.7 implies that for every $p \in \partial\mathbb{F}$ there is at most one point $z \in \partial\Gamma$ for which $L(T_z)$ contains a leaf of the form (p,q).

Suppose now that $\deg(y) = m \geq 2$, so that $(\partial\iota)^{-1}(y) = \{p_1,\dots,p_m\} \subseteq \partial\mathbb{F}$ consists of $m \geq 2$ distinct points. By Theorem 4.6 and Theorem 5.2, we find that for each pair $1 \leq i < j \leq m$ there is some $z_{ij} \in \partial\Gamma$ so that $(p_i,p_j) \in \Lambda_{z_{ij}} = L(T_{z_{ij}})$. The above observation (regarding Proposition 3.7 and the injectivity of κ) shows there is in fact a unique such $z \in \partial\Gamma$; hence we have $(p_i,p_j) \in L(T_z)$ for all $1 \leq i < j \leq m$. This proves that for each $1 \leq i \leq m$ the fiber $(\partial\iota)^{-1}(y)$ has the claimed form

$$(\partial\iota)^{-1}(y) = \{p_i\} \cup \{q \in \partial\mathbb{F} \mid (p_i,q) \in L(T_z)\}.$$

Moreover, since $p_1,\dots,p_m$ are all endpoints of leafs of $L(T_z)$, it is now immediate from Definition 3.9 that each point of $(\partial\iota)^{-1}(y)$ is proximal for $L(T_z)$. Finally, by Proposition 3.8 there is a point $c \in \hat{T}_z$ such that $\mathcal{Q}_{T_z}^{-1}(c) = \{p_1,\dots,p_m\} = (\partial\iota)^{-1}(y)$. This concludes the proof of the lemma. ☐

Definition 6.2 (Γ–essential points)**.** A point $y \in \partial E_\Gamma$ is said to be Γ*–essential* if there exists $z \in \partial\Gamma$ such that $\partial\iota(x) = y$ for some $x \in \partial\mathbb{F}$ proximal for $L(T_z)$ (see Definition 3.9). In this case, there is a unique such point $z \in \partial\Gamma$, which we denote $\zeta(y) := z$.

Propositions 3.7 and 3.10 show that a point $x \in \partial\mathbb{F}$ can be proximal for $L(T_z)$ for at most one point $z \in \partial\Gamma$. Thus $\zeta(y)$ is clearly uniquely determined for any Γ–essential point with $\deg(y) = 1$. This together with Lemma 6.1 shows that Definition 6.2 is justified in asserting that $\zeta(y)$ is uniquely determined. We also note that every $y \in \partial E_\Gamma$ with $\deg(y) \geq 2$ is Γ–essential by Lemma 6.1.

6.1. **Bounding the size of fibers of the Cannon–Thurston map.** Using Lemma 6.1, the $\mathcal{Q}$–index theory for very small $\mathbb{R}$–trees now easily gives a uniform bound on the cardinality of any fiber of the Cannon–Thurston map $\partial\iota\colon \partial\mathbb{F} \to \partial E_\Gamma$.

Theorem 6.3. *Let $\Gamma \leq \mathrm{Out}(\mathbb{F})$ be purely atoroidal and convex cocompact, where $\mathbb{F}$ is a free group of finite rank at least 3, and let $\partial\iota\colon \partial\mathbb{F} \to \partial E_\Gamma$ denote the Cannon–Thurston map for the hyperbolic $\mathbb{F}$–extension E_Γ. Then for every $y \in \partial E_\Gamma$, the degree $\deg(y) = \#\big((\partial\iota)^{-1}(y)\big)$ of the fiber over y satisfies*

$$1 \leq \deg(y) \leq 2\,\mathrm{rank}(\mathbb{F}).$$

In particular, the fibers Cannon–Thurston map are all finite and of uniformly bounded size.

Proof. Fix any point $y \in \partial E_\Gamma$. Since the Cannon–Thurston map is surjective, we clearly have $(\partial\iota)^{-1}(y) \neq \varnothing$. Thus $\deg(y) \geq 1$. If $\deg(y) = 1$ there is nothing to prove, so assume $\deg(y) = m \geq 2$. By Lemma 6.1, there exists a free arational tree $T_z \in \overline{\mathrm{cv}}$ and a point $c \in \hat{T}_z$ so that $(\partial\iota)^{-1}(y) = \mathcal{Q}_{T_z}^{-1}(c)$. Theorem 3.13 then gives

$$m - 2 = \mathrm{ind}_{\mathcal{Q}}(c) \leq \mathrm{ind}_{\mathcal{Q}}(T_z) \leq 2\,\mathrm{rank}(\mathbb{F}) - 2. \qquad \square$$

6.2. **Rational points and the Cannon–Thurston map.** A point in the boundary ∂G of a word-hyperbolic group G is said to be *rational* if it is equal to the limit $g^\infty := \lim_{n\to\infty} g^n$ in $G \cup \partial G$ for some infinite order element $g \in G$. A point in ∂G is *irrational* if it is not rational. Our next result analyzes the fibers of the Cannon–Thurston map $\partial\iota$ over rational points of $\partial\mathbb{F}$ and ∂E_Γ.

Theorem 6.4. *Suppose that $1 \to \mathbb{F} \to E_\Gamma \to \Gamma \to 1$ is a hyperbolic extension with $\Gamma \leq \mathrm{Out}(\mathbb{F})$ convex cocompact. Consider a rational point $g^\infty \in \partial E_\Gamma$, where $g \in E_\Gamma$ has infinite order.*

1. *Suppose that g^k is equal to $w \in \mathbb{F} \lhd E_\Gamma$ for some $k \geq 1$ (i.e., g projects to a finite order element of Γ). Then $(\partial\iota)^{-1}(g^\infty) = \{w^\infty\} \subset \partial\mathbb{F}$ and so $\deg(g^\infty) = 1$.*
2. *Suppose that g projects to an infinite-order element $\phi \in \Gamma$. Then there exists $k \geq 1$ such that the automorphism $\Psi \in \mathrm{Aut}(\mathbb{F})$ given by $\Psi(w) = g^k w g^{-k}$ is forward rotationless (in the sense of* [FH, CH1]*) and its set $\mathrm{att}(\Psi)$ of attracting fixed points in $\partial\mathbb{F}$ is exactly $\mathrm{att}(\Psi) = (\partial\iota)^{-1}(g^\infty)$. Moreover, g^∞ is Γ–essential and $\zeta(g^\infty) = \phi^\infty$.*

Proof. First suppose $g^k = w \in \mathbb{F}$ for some $k \geq 1$. By continuity and $\mathbb{F}$–equivariance of the Cannon–Thurston map, it is immediate that $\partial\iota$ sends $w^\infty \in \partial\mathbb{F}$ to $g^\infty \in \partial E_\Gamma$ (note that $(g^k)^\infty = g^\infty$ in ∂E_Γ). Thus $\{w^\infty\} \subset (\partial\iota)^{-1}(g^\infty)$. Since for every $z \in \partial\Gamma$, T_z is a free arational tree, there do not exist $z \in \partial\Gamma$ and $p \in \partial\mathbb{F}$ such that $(p, w^\infty) \in L(T_z)$. Therefore Theorem 4.6 and Theorem 5.2 imply that $(\partial\iota)^{-1}(g^\infty) \subset \{w^\infty\}$ and part (1) is verified.

Now suppose that g projects to an infinite order element $\phi \in \Gamma$. As explained in Section 3.7, we may choose $k \geq 1$ so that the automorphism $\Psi \in \mathrm{Aut}(\mathbb{F})$ given by $w \mapsto g^k w g^{-k}$ is forward rotationless. Let $p \in att(\Psi)$ be a locally attracting fixed point for the left action of Ψ on $\partial\mathbb{F}$. By Corollary 3.15, p is a proximal point for $L(T_\phi)$, and Corollary 5.7 moreover shows that $T_\phi = T_z$ for the point $z := \phi^\infty \in \partial\Gamma$.

Recall that by Proposition 2.2 the map $\partial\iota \colon \partial\mathbb{F} \to \partial E_\Gamma$ is E_Γ–equivariant, and that g^k acts on $\partial\mathbb{F}$ by Ψ, that is $g^k q = \Psi(q)$ for every $q \in \partial\mathbb{F}$. Since $p = g^k p$ is a local attractor for Ψ, the E_Γ–equivariance and continuity of $\partial\iota$ ensure that $\partial\iota(p) = g^k \partial\iota(p)$ is a local attractor for the action of g^k on ∂E_Γ. Since g is an element of infinite order in a word-hyperbolic group E_Γ, g^∞ is the unique local attractor for the action of g on ∂E_Γ. Thus we may conclude that in fact $\partial\iota(p) = g^\infty$. This proves that $att(\Psi) \subseteq (\partial\iota)^{-1}(g^\infty)$. Since p is proximal for $L(T_z)$, we also see that g^∞ is Γ–essential and that $\zeta(g^\infty) = z = \phi^\infty$, as claimed.

Now, if $\deg(g^\infty) = 1$, it follows that $\#att(\Psi) \leq 1$ so that $att(\Psi) = \{p\} = (\partial\iota)^{-1}(g^\infty)$, as required. On the other hand, if $\deg(g^\infty) \geq 2$, then Lemma 6.1 provides a point $c \in T_\phi$ so that $att(\Psi) \subset (\partial\iota)^{-1}(g^\infty) = \mathcal{Q}_{T_\phi}^{-1}(c)$. Part (2) of Proposition 3.14 then ensures that $(\partial\iota)^{-1}(g^\infty) = att(\Psi)$. Thus claim (2) is verified. $\square$

Kapovich and Lustig showed [KL5] that for the Cannon–Thurston map $\partial\mathbb{F} \to \partial E_{\langle\phi\rangle}$ associated to a cyclic group generated by an atoroidal fully irreducible $\phi \in \mathrm{Out}(\mathbb{F})$, every point $y \in \partial E_{\langle\phi\rangle}$ with $\deg(y) \geq 3$ is rational and has the form $y = g^\infty$ for some $g \in E_{\langle\phi\rangle} - \mathbb{F}$. (Note that here $\partial\langle\phi\rangle$ consists of two points $\phi^{\pm\infty}$, both of which are rational.) Here we show that this result need not hold in the general setting of convex cocompact subgroups. Rather, we find that the Cannon–Thurston map, via the assignment $y \mapsto \zeta(y)$ for Γ–essential points, detects the following relationship between rationality/irrationality in ∂E_Γ and $\partial\Gamma$.

Theorem 6.5. *Suppose that $1 \to \mathbb{F} \to E_\Gamma \to \Gamma \to 1$ is a hyperbolic extension with $\Gamma \leq \mathrm{Out}(\mathbb{F})$ convex cocompact. Then the following hold:*

1. *If $y \in \partial E_\Gamma$ has $\deg(y) \geq 3$ and $\zeta(y) \in \partial\Gamma$ is rational, then y is rational.*

(2) *If* $y \in \partial E_\Gamma$ *has* $\deg(y) \geq 2$ *and* $\zeta(y) \in \partial\Gamma$ *is irrational, then* y *is irrational.*

Our argument for part (1) of the above theorem is similar to the proof of part (3) of Theorem 5.5 in [KL5]. The proof is included here for completeness.

Proof. Suppose first that $y \in \partial E_\Gamma$ is such that $\deg(y) \geq 3$ and $\zeta(y) \in \partial\Gamma$ is rational. Thus $\zeta(y) = \phi^\infty$ for some atoroidal fully irreducible $\phi \in \Gamma$. Then $(\partial\iota)^{-1}(y) = \mathcal{Q}_{T_\phi}^{-1}(x)$ for some $x \in \overline{T_\phi}$. Let $k \geq 1$ be such that $\psi = \phi^k$ is FR. Note that $T_\phi = T_\psi$. Choose some homothety H of $\overline{T}_\phi$. Equation (4) in Section 3.7 implies that H acts on $\mathbb{F}$–orbits of points of $\overline{T}_\phi$. Since there are only finitely many $\mathbb{F}$–orbits of points of $\overline{T}_\phi$ with $\mathcal{Q}_{T_\phi}$–preimage of cardinality ≥ 3 (Proposition 3.11), some positive power of H preserves every such orbit and, in particular, preserves the orbit of x. Thus, after replacing k by kt for some integer $t \geq 1$, we may assume that $Hx = wx$ for some $w \in \mathbb{F}$. Then $w^{-1}Hx = x$. The homothety $H_1 = w^{-1}H$ represents some $\Psi \in \mathrm{Aut}(\mathbb{F})$ whose outer automorphism class is ψ. Moreover, x is the center of the homothety H_1. Therefore by Proposition 3.14 it follows that $\mathcal{Q}_{T_\phi}^{-1}(x) = att(\Psi)$. Thus $(\partial\iota)^{-1}(y) = att(\Psi)$.

Choose $g \in E_\Gamma$ so that the automorphism $h \mapsto ghg^{-1}$ of $\mathbb{F}$ is exactly Ψ. Note that g projects to $\psi \in \Gamma$ and thus g has infinite order in E_Γ. Let $p \in att(\Psi)$. We have $p = \Psi(p) = gp$ and $\partial\iota(p) = y$. By E_Γ–equivariance of $\partial\iota$ it follows that $gy = y$. Since g is an element of infinite order in a word-hyperbolic group E_Γ, it follows that $y = g^{\pm\infty}$ is rational. This proves claim (1).

Next suppose that $\deg(y) \geq 2$ and that $\zeta(y)$ is irrational. Assuming, on the contrary, that y is rational, we have $y = g^\infty$ for some non-torsion element $g \in E_\Gamma$. Since $\deg(y) \geq 2$, Theorem 6.4 (1) implies that g projects to an element of infinite order ϕ of Γ. But then $\zeta(y) = \phi^\infty$ is rational by Theorem 6.4 (2), contradicting the assumption that $\zeta(y)$ was irrational. Therefore y is indeed rational and claim (2) holds. □

6.3. **Conical limit points.** Recall that every non-elementary subgroup of a word-hyperbolic group G acts as a convergence group on the Gromov boundary ∂G. If a group H acts as a convergence group on a compact metrizable space Z, a point $z \in Z$ is called a *conical limit point* for the action of H on Z if there exist an infinite sequence h_n of distinct elements of H and a pair of distinct points $z_-, z_+ \in Z$ such that $\lim_{n\to\infty} h_n z = z_+$ and that $(h_n|_{Z\setminus\{z\}})_n$ converges uniformly on compact subsets to the constant map $c_{z_-} : Z \setminus \{z\} \to Z$ sending $Z \setminus \{z\}$ to z_-. It is also known that if $H \leq G$ is a non-elementary subgroup of a word-hyperbolic group G, then $z \in \partial G$ is a conical limit point for the action of H on ∂G if and only if there exists an infinite sequence of distinct elements $h_n \in H$ such that all h_n lie in a bounded Hausdorff neighborhood of a geodesic ray from 1 to z in the Cayley graph of G. We refer the reader to [JKLO] for more details and background regarding conical limit points.

Theorem 6.6. *Let* $\Gamma \leq \mathrm{Out}(\mathbb{F})$ *be purely atoroidal and convex cocompact. If* $y \in \partial E_\Gamma$ *is* Γ*–essential, then* y *is not a conical limit point for the action of* $\mathbb{F}$ *on* ∂E_Γ*. In particular, if* $\deg(y) \geq 2$ *or if* $y = g^\infty$ *for some* $g \in E_\Gamma$ *projecting to an infinite-order element of* Γ*, then* y *is not a conical limit point for the action of* $\mathbb{F}$*.*

Proof. Choose a free basis X of $\mathbb{F}$. Suppose that $y \in \partial E_\Gamma$ is Γ–essential, and let $z = \zeta(y) \in \partial\Gamma$ so that $y = \partial\iota(p)$ for some $p \in \partial\mathbb{F}$ proximal to $L(T_z) = \Lambda_z$. Then every freely reduced word over $X^{\pm 1}$ which occurs infinitely many times in the geodesic ray from 1 to p in $\mathrm{Cay}(\mathbb{F}, X)$ is a leaf-segment for $L(T_z) = \Lambda_z$. Therefore [JKLO, Theorem B] implies that $y = \partial\iota(p)$ is not a conical limit point for the action of $\mathbb{F}$ on ∂E_Γ, as claimed. The remaining assertions now follow from Lemma 6.1 and Theorem 6.4. □

7. Discontinuity of $z \in \partial\Gamma \mapsto \Lambda_z \in \mathcal{L}(\mathbb{F})$

In this section, we answer a question of Mahan Mitra using our hyperbolic extension E_Γ. As stipulated in Convention 5.1, our fixed finitely generated subgroup $\Gamma \leq \mathrm{Out}(\mathbb{F})$ has a quasi-isometric orbit map into the free factor complex and gives rise to an exact sequence of word-hyperbolic groups

$$1 \longrightarrow \mathbb{F} \longrightarrow E_\Gamma \longrightarrow \Gamma \longrightarrow 1.$$

Thus each point $z \in \partial\Gamma$ has an associated ending lamination Λ_z (Definition 4.3) and we consider the map $F\colon \partial\Gamma \to \mathcal{L}(\mathbb{F})$ defined by $F(z) = \Lambda_z$. Here $\mathcal{L}(\mathbb{F})$ is the set of laminations equipped with the Chabauty topology (Definition 3.1). In his work on Cannon–Thurston maps for normal subgroups of hyperbolic groups, Mitra asked whether this map F is continuous. We answer this question by producing an explicit example for which $F\colon \partial\Gamma \to \mathcal{L}(\mathbb{F})$ is not continuous. This is done in Example 7.5.

Before turning to this example, we establish a "subconvergence" property for the map $F\colon \partial\Gamma \to \mathcal{L}(\mathbb{F})$. This is the strongest positive result that one can give about continuity with respect to the Chabauty topology on $\mathcal{L}(\mathbb{F})$. Recall that for a lamination $L \in \mathcal{L}(\mathbb{F})$ the notation L' denotes the set of accumulation points of L, in the usual topological sense.

Proposition 7.1. *Let $\Gamma \le \mathrm{Out}(\mathbb{F})$ be purely atoroidal and convex cocompact, and let $\Lambda_z \in \mathcal{L}(\mathbb{F})$ denote the ending lamination associated to $z \in \partial\Gamma$. Then for any sequence z_i in $\partial\Gamma$ converging to z and any subsequence limit L of the corresponding sequence Λ_{z_i} in $\mathcal{L}(\mathbb{F})$, we have*

$$\Lambda'_z \subset L \subset \Lambda_z.$$

Before proving Proposition 7.1, we first recall a characterization of convergence in the Chabauty topology. Let X be a locally compact metric space and let $C(X)$ be the space of closed subsets of X equipped with the Chabauty topology. Recall that $C(X)$ is compact. The following lemma is well known; see [CME].

Lemma 7.2 (Chabauty convergence)**.** *For a locally compact metric space X, a sequence C_i converges to C in $C(X)$ if and only if the following conditions are satisfied:*

1. *For each $x_{i_k} \in C_{i_k}$, whenever $x_{i_k} \to x$ in X as $k \to \infty$ it follows that $x \in C$.*
2. *For each $x \in C$, there is are $x_i \in C_i$ with $x_i \to x$ in X as $i \to \infty$.*

As "weak evidence" towards continuity of the map F, Mitra proves the following proposition [Mit1], which essentially amounts to verifying (1) in Lemma 7.2.

Proposition 7.3 (Proposition 5.3 of [Mit1])**.** *If $z_i \to z$ in $\partial\Gamma$ and if $(p_i, q_i) \in \Lambda_{z_i}$ converge to (p, q) in $\partial^2\mathbb{F}$, then $(p, q) \in \Lambda_z$.*

Remark 7.4 (Semi-continuity of the map $T \mapsto L(T)$)**.** In [CHL4], the authors remark that if $(T_i)_{i\ge 0}$ is a sequence of trees in $\overline{\mathrm{cv}}$ converging to a tree T, then any subsequence limit L of the corresponding sequence of dual laminations $(L(T_i))_{i\ge 0}$ in $\mathcal{L}(\mathbb{F})$ is contained in $L(T)$. They elaborate this statement in [CHL1, Proposition 1.1]. Combining this general fact with Theorem 5.2 gives an alternative proof of Proposition 7.3.

We now turn to the proof of Proposition 7.1.

Proof of Proposition 7.1. Suppose that $z_i \to z$ in $\partial\Gamma$. Then by Proposition 7.3 and Lemma 7.2, if L is any subsequential limit of Λ_{z_i} in the Chabauty topology it follows that $L \subset \Lambda_z$. By Theorem 5.2, $\Lambda_z = L(T_z)$ for the free and arational tree T_z; we also know that $L(T_z)$ is the diagonal closure of its unique minimal sublamination $L'(T_z)$ by [CHR, Theorem A]. In particular, we must have that $L \supset L'(T_z) = \Lambda'_z$. Hence, $\Lambda'_z \subset L \subset \Lambda_z$, as required. □

We conclude this section by producing an example of a hyperbolic extensions E_Γ for which $F\colon \partial\Gamma \to \mathcal{L}(\mathbb{F})$ is not continuous. Before explaining this example, we briefly recall some facts related to the index theory of free group automorphisms. We refer the reader to [CH1, KL4, CH2, CHR] for more details.

Let $\phi \in \mathrm{Out}(\mathbb{F})$ be an atoroidal fully irreducible element and let $h\colon G \to G$ be a train track representative of ϕ (thus h is necessarily expanding and irreducible). By replacing h with a sufficiently large positive power and possibly subdividing G, we may further assume the following: that the endpoints of all INPs in G (if any are present) are vertices of G, that every periodic vertex of G is fixed by h, that every periodic direction in G has period 1 and that every periodic INP (if any are present) in G has period 1. Here, the acronym INP stands for irreducible Nielsen path; see [BH, BFH, FH] for background. In the discussion below, if $(p, q) \in L$ for some algebraic lamination L on $\mathbb{F}$, we will often refer to a geodesic $\mathfrak{l}$ from p to q in $\tilde{G}$ as a leaf of L.

Recall that the *Bestvina-Feighn-Handel lamination* $L_{BFH}(\phi)$ is an algebraic lamination on $\mathbb{F}$ consisting of all $(p, q) \in \partial^2\mathbb{F}$ such that for every finite subpath $\tilde{\gamma}$ of the geodesic in $\tilde{\Gamma}$ connecting p to q, the projection γ of $\tilde{\gamma}$ to G is a subpath of $h^n(e)$ for some $n \geq 1$ and some (oriented) edge e of G. If v is a periodic vertex of G and e is an edge starting with v defining a periodic direction at v (so that $h(e)$ starts with e), then e determines a semi-infinite reduced edge-path ρ_e in G called the *eigenray* of h corresponding to v. Namely, ρ_e is defined as the path such that for every $n \geq 1$, $h^n(e)$ is an initial segment of ρ_e. It is known [KL4] that $L_{BFH}(\phi) \subseteq L(T_\phi)$ is the unique minimal sublamination of $L(T_\phi)$; that is, $L_{BFH}(\phi)$ is the unique minimal (with respect to inclusion) nonempty subset of $L(T_\phi)$ which is itself an algebraic lamination on $\mathbb{F}$. It is also known that $L(T_\phi)$ is the "transitive closure" of $L_{BFH}(\phi)$; that is, $L(T_\phi)$ is the smallest diagonally closed (in the sense defined in Section 3.1) subset of $\partial^2\mathbb{F}$ which contains $L_{BFH}(\phi)$ and is itself a lamination. Moreover, $L(T_\phi) \setminus L_{BFH}(\phi)$ consists of finitely many $\mathbb{F}$–orbits of points of $\partial^2\mathbb{F}$ called *diagonal leaves* of $L(T_\phi)$, and [KL4] gives a precise description of these diagonal leaves in terms of the train track h: If v is a periodic vertex and e, e' are distinct periodic edge of G with origin v, then any lift to $\tilde{\Gamma}$ of the biinfinite path $\rho_e^{-1}\rho_{e'}$ is a leaf of $L(T_\phi)$; such a leaf is called a *special leaf*. Some of the special leaves already belong to $L_{BFH}(\phi)$ (this happens precisely when the turn e, e' is "taken" by h). Special leaves that do not belong to $L_{BFH}(\phi)$ are necessarily diagonal. If h has no periodic INPs, then all diagonal leaves of $L(T_\phi)$ arise in this way; that is, every diagonal leaf is special. If h has some periodic INPs, then $L(T_\phi)$ admits diagonal leaves of additional kind, but their precise description is not needed here (see [KL4] for details).

Example 7.5 (Discontinuity of $F\colon \partial\Gamma \to \mathcal{L}(\mathbb{F})$)**.** We now construct an example of a purely atoroidal convex cocompact subgroup $\Gamma \leq \mathrm{Out}(\mathbb{F})$, with $\mathrm{rank}(\mathbb{F}) = 3$, such that the map $\partial\Gamma \to \mathcal{L}(\mathbb{F})$ given by $z \mapsto \Lambda_z$ is not continuous with respect to the Chabauty topology on $\mathcal{L}(\mathbb{F})$.

Suppose that we are given automorphisms ϕ and ψ of $\mathbb{F} = F(a, b, c)$, the free group of rank 3, with the following properties:

(1) ϕ and ψ are atoroidal and fully irreducible.
(2) ϕ and ψ are positive with respect to the basis $\{a, b, c\}$. Thus we can represent ϕ and ψ by train track maps on the rose R_3 with a single vertex v and petals corresponding to a, b, c; we denote these train track maps by f and g accordingly. We further assume that we have replaced f and g by appropriate positive powers so that for each of f, g every periodic vertex of R_3 is fixed, that every periodic direction in R_3 has period 1, and that every periodic INP in R_3 (if any are present) has period 1.
(3) For the map f the directions corresponding to the three edges of R_3 labelled a, b, c are periodic.
(4) The map g has 4 periodic directions at v, given by the edges labelled by a, c, a^{-1}, b^{-1}. Moreover, g has no periodic INPs.

At the end of this example, we will give references for where one can find automorphisms satisfying these conditions.

By [BFH, KL2, DT1], we may replace ϕ and ψ by further positive powers such that $\Gamma = \langle \phi, \psi \rangle$ is a purely atoroidal, convex cocompact subgroup of $\mathrm{Out}(\mathbb{F})$. Hence, the corresponding extension E_Γ is hyperbolic. We will show that the map $F\colon \partial\Gamma \to \mathcal{L}(\mathbb{F})$ defined by $F(z) = \Lambda_z$ is not continuous.

For a leaf $\mathfrak{l}$ of a lamination $L \in \mathcal{L}(\mathbb{F})$, we say that $\mathfrak{l}$ is *positive* (correspondingly *negative*) if $\mathfrak{l}$ is labelled by a positive (correspondingly negative) bi-infinite word in $F(a, b, c)$, and we say that $\mathfrak{l}$ is *mixed* if it is neither positive nor negative. Note that, since the automorphism ψ is positive, the definition of $L_{BFH}(\psi)$ implies that every leaf of $L_{BFH}(\psi)$ is, up to a flip, labelled by a positive biinfinite word in $F(a,b,c)$. Also recall that, as discussed above, $L_{BFH}(\psi)$ is the unique minimal sublamination of $L(T_\psi)$. Since g has no periodic INPs, every mixed leaf of $L(T_\psi)$ is special. Hence, each mixed leaf of $L(T_\psi)$ is, up to the $\mathbb{F}$–action and the flip, labelled by a word of the form $\rho_1^{-1}\rho_2$ where ρ_1, ρ_2 are two eigenrays of g corresponding to two distinct periodic directions at v.

We establish following: (i) there are exactly 2 mixed diagonal leaves of $L(T_\psi)$, up to the $\mathbb{F}$–action and the flip, (ii) if $\phi^n L(T_\psi)$ converges to L in $\mathcal{L}(\mathbb{F})$ then L contains at most 2 mixed leaves, up to the $\mathbb{F}$–action and the flip; (iii) the lamination $L(T_\phi)$ contains at least 3 distinct mixed diagonal leaves, up to the $\mathbb{F}$–action and the flip.

To see (i), note that since we assumed that g has no periodic INPs, mixed leaves of $L(T_\psi)$ are leaves of $L(T_\psi) \setminus L_{BFH}(\psi)$ and hence are diagonal and special for $L(T_\psi)$. Recall that g has exactly four periodic directions at v, namely a, c, a^{-1}, b^{-1}. Thus g has 4 eigenrays starting at v: positive eigenrays ρ_a, ρ_c and negative eigenrays $\rho_{a^{-1}}$, $\rho_{b^{-1}}$. Up to a flip, every mixed leaf of $L(T_\psi)$ is then labeled by either $(\rho_a)^{-1}\rho_c$ or $(\rho_{a^{-1}})^{-1}\rho_{b^{-1}}$. Thus (i) is verified.

A similar argument can be used to prove (ii), although a bit of care is needed here in using the definition of the Chabauty topology on $\mathcal{L}(\mathbb{F})$. Suppose $L \in \mathcal{L}(\mathbb{F})$ is the limit of $\phi^n L(T_\psi)$ as $n \to \infty$. Any mixed leaf $\mathfrak{l}$ of L, up to a flip, is labelled by a word of the form $W^{-1}Z$, where W and Z are each positive rays from the identity in $F(a,b,c)$ starting with distinct symbols. Since $\lim_{n\to\infty} \phi^n L(T_\psi) = L$ in the Chabauty topology, and since ϕ is a positive automorphism, every such mixed leaf $\mathfrak{l}$ of L must be a subsequential limit of mixed leaves of $\phi^n L(T_\psi)$, which are labelled by words of the form $\phi^n\left((\rho_a)^{-1}\rho_c\right)$ or $\phi^n\left((\rho_{a^{-1}})^{-1}\rho_{b^{-1}}\right)$. Finally note that the assumptions on f imply that if $n_i, m_i \to \infty$ are two sequences of indices such that some mixed leaves of $\phi^{n_i} L(T_\psi)$ labelled by words of the form $\phi^{n_i}\left((\rho_a)^{-1}\rho_c\right)$ converge to a mixed leaf $\mathfrak{l}_1$ of L and that some leaves of $\phi^{m_i} L(T_\psi)$ labelled by words of the form $\phi^{m_i}\left((\rho_a)^{-1}\rho_c\right)$ converge to a mixed leaf $\mathfrak{l}_2$ of L, then the leaves $\mathfrak{l}_1, \mathfrak{l}_2$ have the same label (up to a shift). The same holds when $(\rho_a)^{-1}\rho_c$ is replaced by $(\rho_{a^{-1}})^{-1}\rho_{b^{-1}}$. It follows that, up to the $\mathbb{F}$–action and the flip, there are at most 2 mixed leaves in L, and (ii) is verified.

Finally, we observe (iii). Let r_a, r_b, r_c be the f–eigenrays in R_3 corresponding to the f–periodic directions a, b, c at v. Thus r_a, r_b, r_c are positive semi-infinite words. Then there exist special mixed leaves in $L(T_\phi)$ labelled by $(r_a)^{-1}r_b$, $(r_a)^{-1}r_c$, $(r_b)^{-1}r_c$. These leaves are distinct, up to the $\mathbb{F}$–action and the flip. Thus (iii) is verified.

Now let $\phi^\infty = \lim_{n\to\infty} \phi^n \in \partial\Gamma$ and $\psi^\infty = \lim_{n\to\infty} \psi^n \in \partial\Gamma$. We know that the orbit map $\Gamma \to \mathcal{F}$ induces an embedding $\partial\Gamma \to \partial\mathcal{F}$ which takes ϕ^∞ to T_ϕ and ψ^∞ to T_ψ. We argue that $F\colon \partial\Gamma \to \mathcal{L}(\mathbb{F})$ is not continuous by contradiction. Indeed suppose that F is continuous. Then by Theorem 5.2

$$\begin{aligned}
\lim_{n\to\infty} \phi^n L(T_\psi) &= \lim_{n\to\infty} \phi^n \Lambda_{\psi^\infty} \\
&= \lim_{n\to\infty} \phi^n F(\psi^\infty) \\
&= \lim_{n\to\infty} F(\phi^n \psi^\infty) \\
&= F(\phi^\infty) = \Lambda_{\phi^\infty} = L(T_\phi),
\end{aligned}$$

where convergence in $\mathcal{L}(\mathbb{F})$ is with respect to the Chabauty topology. Together with (ii), the fact that $\lim_{n\to\infty} \phi^n L(T_\psi) = L(T_\phi)$ implies that, up to the $\mathbb{F}$–action and the flip, there are at most 2 distinct mixed leaves in $L(T_\phi)$. However, this contradicts (iii). Thus F is not continuous.

To complete the example, it only remains to give an example of automorphisms ϕ and ψ satisfying conditions (1)–(4). The automorphism ϕ can be taken to be the automorphism α_3 constructed by Jağer and Lustig in [JL]. This automorphism is given by $f(a) = abc$, $f(b) = bab$, and $f(c) = cabc$, and each of the required properties is verified by Jağer and Lustig. For the automorphism ψ, we may take (a rotationless power of) the automorphism constructed by Pfaff in Example 3.2 of [Pfa]. This is the automorphism $g(a) = cab$, $g(b) = ca$, and $g(c) = acab$, and the required properties are established by Pfaff. This completes the example.

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