PRACTICE MIDTERM

- Each question is for 6 pts.
- Total points: 30

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1) True or false. Provide an explanation for your answer as well.
i) The sequence of functions $f_n(x) = x^n(1-x)$ converges uniformly to 0 on the interval $[0,1]$.
ii) The laplace Neumann boundary value problem on the interval $[0,1]$

$$\begin{aligned} u''(x) &= 0\,, \\ u_x(0) &= f\,, \\ u_x(1) &= g\,, \end{aligned}$$

is well posed if $f = g$ and $\int_0^1 u(x)\,dx = 0$.
iii) The differential operator $\mathcal{L}[u] = -u''$ defined on the interval $x \in [0,1]$ always has positive eigenvalues

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2) Solve the following PDE

$$\begin{aligned} \partial_{tt}u = c^2\partial_{xx}u \quad 0 < x < \infty\,, \quad 0 < t\,, \\ u(0,t) = t\,, \quad 0 < t\,, \\ u(x,0) = \sin(x)\,, \quad 0 < x < \infty\,, \\ u_t(x,0) = x\,, \quad 0 < x < \infty\,. \end{aligned}$$

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3) Compute all separation of variables solutions of

$$\begin{aligned} u_t = u_{xx} + 4u\,, \quad 0 < x < 1\,, 0 < t \\ u(0,t) = 0 \\ u_x(1,t) = 0 \\ u(x,0) = \phi(x)\,. \end{aligned}$$

Find the solution if the initial data is given by

$$\phi(x) = \sin\left(\frac{\pi}{2}x + 2\pi\right) + 2\sin\left(\frac{\pi}{2}x + 8\pi\right)$$

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4) Prove that, among all possible dimensions, only in three dimensions can one have distortionless spherical wave propagation with attenuation. This means the following. A spherical wave in $n-$dimensional space satisfies the PDE

$$u_{tt} = c^2\left(u_{rr} + \frac{n-1}{r}u_r\right)\,,$$

where r is the spherical coordinate. Consider such a wave that has the special form

$$u(r,t) = \alpha(r)\,f(t-\beta(r))\,,$$

where $\alpha(r)$ is the attenuation and $\beta(r)$ is the delay. The question is whether such solutions exist for "arbitrary" functions f.
a) Plug the special form into the PDE to get an ODE for f.
b) Set the coefficients of f'', f' and f equal to 0.
c) Solve the ODEs to see that $n = 1$ or $n = 3$
d) If $n = 1$, show that $\alpha(r)$ is a constant.

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5) Obtain a general solution to the following PDEs and sketch the characteristics in both cases
i)

$$au_x + bu_y + cu = 0$$

ii)

$$u_x + u_y = 1$$