PRACTICE PROBLEM SET 5

- **Chap 5**
- Questions are either directly from the text or a small variation of a problem in the text.
- Collaboration is okay, but final submission must be written individually. Mention all collaborators on your submission.
- The terms in the bracket indicate the problem number from the text.

Section 9.2

1) (Prob 6, Pg 240) a) Let S be the spehere of center $\boldsymbol{x}$ and radius R. What is the surface area of $S \cap \{|\boldsymbol{x}| < \rho\}$, the portion of S that lies within the spehere of center 0 and radius ρ?

b) Solve the wave equation in three dimensions for $t > 0$ with the initial conditions $\phi(\boldsymbol{x}) = 0$, $\psi(\boldsymbol{x}) = A$ for $|\boldsymbol{x}| < \rho$, and $\psi(\boldsymbol{x}) = 0$ for $|\boldsymbol{x}| > \rho$, where A is a constant.

c) Let $|\boldsymbol{x}_0| < \rho$. Ride the wave along a light ray emanating from $(\boldsymbol{x}_0, 0)$. That is, look at $u(\boldsymbol{x}_0 + t\boldsymbol{v}, t)$, where $|\boldsymbol{v}| = c$. Prove that

$$t \cdot u(\boldsymbol{x}_0 + t\boldsymbol{v}, t) \text{converges as } t \to \infty .$$

Solution:

a)

$$S \cap \{|\boldsymbol{x}| < \rho\} = \begin{cases} \frac{\pi R}{|\boldsymbol{x}|}\left(\rho^2 - (|\boldsymbol{x}| - R)^2\right) & |\rho - R| \le |\boldsymbol{x}| < \rho + R \\ 0 & |\boldsymbol{x}| > \rho + R \\ \min\{4\pi\rho^2, 4\pi R^2\} & |\boldsymbol{x}| \le |\rho - R| \end{cases}$$

b)

$$u(\boldsymbol{x}, t) = \begin{cases} At & |\boldsymbol{x}| \le \rho - ct \\ \frac{A}{4c|\boldsymbol{x}|}\left(\rho^2 - (|\boldsymbol{x}| - ct)^2\right) & |\rho - ct| \le |\boldsymbol{x}| \le \rho + ct \end{cases} .$$

c)

$$\lim_{t\to\infty} t \cdot u(\boldsymbol{x}_0 + t\boldsymbol{v}, t) = \lim_{t\to\infty} \frac{At}{4c\,|\boldsymbol{x}_0 + t\boldsymbol{v}|}\left(\rho^2 - (|\boldsymbol{x}_0 + t\boldsymbol{v}| - ct)^2\right) = \frac{A}{4c^2}\left(\rho^2 - \frac{(\boldsymbol{x}_0, v)^2}{c^2}\right)$$

2) (Prob 13, Pg 241) Solve the wave equation in the half-space $\{(x, y, z, t) : z > 0\}$with the Neumann condition $\frac{\partial u}{\partial z} = 0$ on $z = 0$ and with initial data $\phi(x, y, z) \equiv 0$ and general $\psi(x, y, z)$.

Just use the even extension of ψ about the z axis. Then $\partial_z \psi(x, y, z)$ is an odd function of z. Just think in a one dimensional form.

Using this even extension, the solution is given by

$$u(x, y, z, t) = \frac{1}{4\pi c^2 t} \iint_{x_0^2+y_0^2+z_0^2=c^2t^2} \psi(x + x_0, y + y_0 z + z_0)\, dS$$
$$\partial_z u(x, y, z, t) = \frac{1}{4\pi c^2 t} \iint_{x_0^2+y_0^2+z_0^2=c^2t^2} \partial_z \psi(x + x_0, y + y_0 z + z_0)\, dS$$

Then

$$\begin{aligned} \partial_z u(x, y, 0, t) &= \frac{1}{4\pi c^2 t} \iint_{x_0^2+y_0^2+z_0^2=c^2t^2} \partial_z \psi(x + x_0, y + y_0, z_0)\, dS \\ &= \iint_{z_0=\sqrt{c^2t^2-x_0^2-y_0^2}} \partial_z \psi(x + x_0, y + y_0 z_0)\, dS + \iint_{z_0=-\sqrt{c^2t^2-x_0^2-y_0^2}} \partial_z \psi(x + x_0, y + y_0, z_0)\, dS \\ &= \iint_{z_0=\sqrt{c^2t^2-x_0^2-y_0^2}} \partial_z \psi(x + x_0, y + y_0 z_0)\, dS + \iint_{z_0=\sqrt{c^2t^2-x_0^2-y_0^2}} \partial_z \psi(x + x_0, y + y_0, -z_0)\, dS = 0 \end{aligned}$$

3) (Prob 16, Pg 241) Solve part b) for the same problem in 2 dimensions. Furthermore, compute $u(0, t)$ by computing the integral explicitly and compute the limit of $u(0, t)$ as $t \to \infty$.

Solution:

$\frac{A\rho^2}{2c^2}$

Section 14.1
4) (Prob 5, Pg 389) Solve $u_t + u^2 u_x = 0$ with $u(x,0) = 2 + x$
Solution:

$$x - x_0 = (2 + x_0)^2 t$$

are the chacteristics. The solution is then given by

$$u(t,x) = \left[\sqrt{4tx + 8t + 1} - 1\right] / 2t$$

5) (Prob 10, Pg 389) Solve $u_t + uu_x = 0$ with initial conditions $u(x,0) = 1$ for $x \le 0$, $1 - x$ for $0 \le x \le 1$ and 0 for $x \ge 1$. Solve for all $t \ge 0$, allowing for a shock wave.
Solution:

$$u(t,x) = \begin{cases} 1 & x < t,\, t \le 1 \\ 0 & x > 1,\, t \le 1 \\ \frac{1-x}{1-t} & t < x < 1,\, t \le 1 \\ 1 & x - 1 < \frac{1}{2}(t-1),\, t > 1 \\ 0 & x - 1 > \frac{1}{2}(t-1),\, t > 1 \end{cases}$$

Section 14.3
6) (Prob 4, Pg 400) Find the curve $y = u(x)$ that makes the integral $\int_0^1 \left((u')^2 + xu\right) dx$ sationary subject to the constraints $u(0) = 0$ and $u(1) = 1$.
Solution

$$u(x) = \frac{x^3 + 11x}{12}$$

7) (Prob 7, Pg 401) Show that there are an infinite number of functions that minimize the integral

$$\int_0^2 (y')^2 (1 + y')^2 \quad \text{subject to } y(0) = 1 \text{and } y(2) = 0\,.$$

They are continuous functions with piecewise continuous first derivatives.
Solution:
Keep on alternating the derivative to change between -1 and 0 so that net change in value is 1 over interval length of 2. For example,

$$y' = \begin{cases} 0 & 0 < x < 1 \\ -1 & 1 < x < 2 \end{cases}$$

$$y' = \begin{cases} -1 & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

In general let $y' = -1$ on some subinterval whose length is 1 and 0 on the other subinterval.
8) (Prob 11, Pg 401) If the action $A[u] = \iint \left(u_{xx}^2 - u_t^2\right) dxdt$, show that the Euler-Lagrange equation is the beam equation $u_{tt} + u_{xxxx} = 0$, the equation for a stiff rod.
Solution:

$$\begin{aligned} f(\epsilon) = A[u + \epsilon v] &= \iint \left(u_{xx}^2 + \epsilon^2 v_{xx}^2 + 2u_{xx}\epsilon v_{xx} - u_t^2 - \epsilon^2 v_t^2 - 2\epsilon u_t v_t\right) dxdt \\ f'(0) &= \iint (u_{xx}v_{xx} - u_t v_t)\, dxdt \\ &= \iint (u_{xxxx} + u_{tt})\, v\, dxdt \quad \text{(Integration by parts)} \end{aligned}$$