PRACTICE PROBLEM SET 2

Practice problems:

1) (Sec 2.3, prob 6) Prove the comparison principle for the diffusion equation or heat equation. If u, v are both solutions to the heat equation for $x \in [0,1]$ and $t \in [0,T]$, and if $u \leq v$ for $t = 0$, for $x = 0$ and for $x = 1$. Then $u \leq v$ for all $x \in [0,1]$ and $t \in [0,T]$.

Solution: Follows from the maximum principle. Consider $w = u - v$. By linearity w satisfies the heat equation. Moreover $w \leq 0$ on the boundary. By the maximum principle, $w \leq 0$ everywhere in the domain from which the result follows.

2) (Sec 2.3, prob 8) Consider the diffusion equation for $x \in [0,1]$ with the Robin boundary condition, $u_x(0,t) - a_0 u(0,t) = 0$ and $u_x(1,t) + a_1 u(1,t) = 0$. If $a_0, a_1 > 0$ show that

$$e(t) = \int_0^1 u^2(x,t)\, dx$$

is a decreasing function of time, i.e. energy is lost at the boundary.

Solution:

$$\begin{aligned}
e'(t) &= \int_0^1 2uu_t\, dx \\
&= 2\int_0^1 kuu_{xx}\, dx \\
&= 2kuu_x|_0^1 - 2k\int_0^1 u_x^2\, dx \\
&= -2ka_1 u(1,t)^2 - 2ka_0 u(0,t)^2 - 2k\int_0^1 u_x^2\, dx \\
&\leq 0
\end{aligned}$$

3) (Sec 2.4, prob 1) Solve the diffusion equation with initial condition

$$\phi(x) = \begin{cases} 1 & -2 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

You may express the solution in terms of the erf function defined below:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-p^2}\, dp$$

Solution:

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\phi(y)\ dy \\
&= \frac{1}{\sqrt{4\pi kt}}\int_{-2}^{1} e^{-\frac{(x-y)^2}{4kt}}\phi(y)\ dy \\
&= \frac{1}{\sqrt{\pi}}\int_{\frac{(x-1)}{\sqrt{4kt}}}^{\frac{(x+2)}{\sqrt{4kt}}} e^{-p^2}\ dp \qquad \left(\frac{(x-y)}{\sqrt{4kt}} = p\right) \\
&= \frac{1}{2}\left(\operatorname{erf}\left(\frac{x+2}{\sqrt{4kt}}\right) - \operatorname{erf}\left(\frac{x-1}{\sqrt{4kt}}\right)\right)
\end{aligned}$$

4) (Sec 2.4, Prob 6,7) Compute

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)} dxdy$$

by transforming the integral to polar coordinates. Using the computation above and symmetry arguments, compute

$$\int_{-\infty}^{\infty} e^{-x^2} dx.$$

Using a suitable change of variables, deduce that

$$\int_{-\infty}^{\infty} S(x,t)\ dx = 1 \quad \forall t.$$

Suppose $u(x,t)$ is a solution to the heat equation with initial data $\phi(x)$. Show that

$$\int_{-\infty}^{\infty} |u(x,t)|\, dx \le \int_{-\infty}^{\infty} |\phi(x)|\, dx \quad \forall t.$$

Solution:

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)} dxdy &= \int_{0}^{\infty}\int_{0}^{2\pi} e^{-r^2} r\, dr\, d\theta \\ &= \pi\end{aligned}$$

$$\begin{aligned}\pi &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(x^2+y^2\right)} dxdy \\ &= \int_{-\infty}^{\infty} e^{-x^2}\, dx \int_{-\infty}^{\infty} e^{-y^2}\, dy \\ &= \left(\int_{-\infty}^{\infty} e^{-x^2}\, dx\right)^2 \\ \therefore \int_{-\infty}^{\infty} e^{-x^2}\, dx &= \sqrt{\pi}\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty} S(x,t)\ dx &= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{4kt}}\, dx \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-p^2}\, dp \quad \left(\frac{x}{\sqrt{4kt}} = p\right) \\ &= 1\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{\infty} |u(x,t)|\ dx &= \int_{-\infty}^{\infty} \left|\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\phi(y)\ dy\right|\ dx \\ &\le \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|e^{-\frac{(x-y)^2}{4kt}}\phi(y)\right|\, dy\, dx \quad \left(\left|\int f\right| \le \int |f|\right) \\ &= \int_{-\infty}^{\infty} |\phi(y)|\,\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\, dx\, dy \quad \text{(Switching order of integration)} \\ &= \int_{-\infty}^{\infty} |\phi(y)|\ dy\end{aligned}$$

5) (Sec 2.5, Prob 1) Construct an example to show that there is no maximum principle for the wave equation.
Solution: $\phi(x) = 1$ and $\psi(x) = 1$ for $-1 \le x \le 1$. Then $u\left(0, \frac{1}{c}\right) = 2$.
6) Solve the following heat and wave equation on the half line $0 < x < \infty$ and comment on the results:

$$\begin{aligned} u_t = u_{xx} &\quad u(x,0) = \phi(x) \\ u_{tt} = u_{xx} &\quad u(x,0) = \phi(x) \quad u_t(x,0) = 0\end{aligned}$$

where $\phi(x)$ is the function

$$\phi(x) = \begin{cases} 1 & 1 \le x \le 2 \\ 0 & \text{otherwise}\end{cases}$$

Carefully sketch the solution for the wave equation.
Solution:
Diffusion equation:

$$
\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left(\exp\left(-\frac{(x-y)^2}{4kt} \right) - \exp\left(-\frac{(x+y)^2}{4kt} \right) \right) \phi(y)\ dy \\
&= \frac{1}{\sqrt{4\pi kt}} \int_1^2 \left(\exp\left(-\frac{(x-y)^2}{4kt} \right) - \exp\left(-\frac{(x+y)^2}{4kt} \right) \right) dy \\
&= \frac{1}{2} \left(\operatorname{erf}\left(\frac{x-1}{\sqrt{4kt}} \right) - \operatorname{erf}\left(\frac{x-2}{\sqrt{4kt}} \right) \text{-}\operatorname{erf}\left(\frac{x+2}{\sqrt{4kt}} \right) + \operatorname{erf}\left(\frac{x+1}{\sqrt{4kt}} \right) \right)
\end{aligned}
$$

Wave equation:

Think of the solution as two copies of $-\frac{1}{2}$ supported on both $[-2,-1]$ and two copies of $\frac{1}{2}$ supported on $[1,2]$. Now one of each of this copy moves to the left with speed c and the other copy moves to the right with speed c. Restrict the solution to $x > 0$, to get the final answer.

Additional problem:

1) Maximum principle and Uniqueness for solutions to heat equation on the real line:

Consider the heat equation on the real line:

$$u_t = u_{xx} \quad x \in (-\infty, \infty) \quad t \in (0, T], \tag{1}$$

$$u(x,0) = g(x) \tag{2}$$

Unfortunately, it is known that without additional conditions on u or g, there exist more than one solution to the above equation. For those interested, you should look up Tychonoff solutions to the heat equation. However, let us make a further assumption on the growth of u:

$$|u(x,t)| \leq M e^{a|x|^2} \quad \forall t \in [0,T] \tag{3}$$

Prove that if u satisfies equations 1, 2, and 3, then u satisfies the maximum principle

$$u(x,t) \leq \sup_{-\infty < x < \infty} g(x) \quad \forall x \in (-\infty, \infty), \quad t \in [0,T]. \tag{4}$$

To prove this result fill follow the steps outlined below:

i) Without loss of generality, one may assume that $\sup g < \infty$ and furthermore assume $4aT < 1$. Consider the function

$$v(x,t) = u(x,t) - \mu w(x,t) \quad x \in (-\infty, \infty) \quad t \in [0,T]$$

where

$$w(x,t) = \frac{1}{(T+\epsilon-t)^{\frac{1}{2}}} \exp\left(\frac{|x|^2}{T+\epsilon-t} \right)$$

What initial value problem does $v(x,t)$ satisfy? How do the initial values of $v(x,t)$ compare to the initial values of $u(x,t)$, i.e. what is the relation between $v(x,0)$ and $\sup_y g(y)$

ii) Using the growth condition for $u(x,t)$,show that there exists a sufficiently large R such that

$$v(x,t) \leq \sup_{y \in (-\infty,\infty)} g(y) \quad |x| \geq R, t \in [0,T] \tag{5}$$

iii) Apply the maximum principle for $v(x,t)$ on the finite domain $|x| \leq R, t \in [0,T]$ to conclude that

$$v(x,t) \leq \sup_{y \in (-\infty,\infty)} g(y) \quad x \in (-\infty, \infty), t \in [0,T]$$

iv) The above result was valid for all values of μ. Take the limit $\mu \to 0$ to conclude that u satisfies the maximum principle

v) Use the maximum principle to show that the heat equation coupled with the growth conditions on u has a unique solution.