PRACTICE PROBLEM SET 2

Practice problems:

1) (Sec 2.3, prob 6) Prove the comparison principle for the diffusion equation or heat equation. If u, v are both solutions to the heat equation for $x \in [0,1]$ and $t \in [0,T]$, and if $u \leq v$ for $t = 0$, for $x = 0$ and for $x = 1$. Then $u \leq v$ for all $x \in [0,1]$ and $t \in [0,T]$

2) (Sec 2.3, prob 8) Consider the diffusion equation for $x \in [0,1]$ with the Robin boundary condition, $u_x(0,t) - a_0 u(0,t) = 0$ and $u_x(1,t) + a_1 u(1,t) = 0$. If $a_0, a_1 > 0$ show that

$$e(t) = \int_0^1 u^2(x,t)\, dx$$

is a decreasing function of time, i.e. energy is lost at the boundary.

3) (Sec 2.4, prob 1) Solve the diffusion equation with initial condition

$$\phi(x) = \begin{cases} 1 & -2 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

You may express the solution in terms of the erf function defined below:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\, dp$$

4) (Sec 2.4, Prob 6,7) Compute

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)}\, dx dy$$

by transforming the integral to polar coordinates. Using the computation above and symmetry arguments, compute

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx.$$

Using a suitable change of variables, deduce that

$$\int_{-\infty}^{\infty} S(x,t) = 1 \quad \forall t.$$

Suppose $u(x,t)$ is a solution to the heat equation with initial data $\phi(x)$. Show that

$$\int_{-\infty}^{\infty} |u(x,t)|\, dx \leq \int_{-\infty}^{\infty} |\phi(x)|\, dx \quad \forall t.$$

5) (Sec 2.5, Prob 1) Construct an example to show that there is no maximum principle for the wave equation.

6) Solve the following heat and wave equation on the half line $0 < x < \infty$ and comment on the results:

$$\begin{aligned} u_t &= u_{xx} \quad u(x,0) = \phi(x) \\ u_{tt} &= u_{xx} \quad u(x,0) = \phi(x) \quad u_t(x,0) = 0 \end{aligned}$$

where $\phi(x)$ is the function

$$\phi(x) = \begin{cases} 1 & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Carefully sketch the solution for the wave equation.

Additional problem:

1) Maximum principle and Uniqueness for solutions to heat equation on the real line:
Consider the heat equation on the real line:

$$u_t = u_{xx} \quad x \in (-\infty, \infty) \quad t \in (0,T], \tag{1}$$

$$u(x,0) = g(x) \tag{2}$$

Unfortunately, it is known that without additional conditions on u or g, there exist more than one solution to the above equation. For those interested, you should look up Tychonoff solutions to the heat equation. However, let us make a further assumption on the growth of u:

(3) $$|u(x,t)| \leq Me^{a|x|^2} \quad \forall t \in [0,T]$$

Prove that if u satisfies equations 1, 2, and 3, then u satisfies the maximum principle

(4) $$u(x,t) \leq \sup_{-\infty<x<\infty} g(x) \quad \forall x \in (-\infty,\infty), \quad t \in [0,T].$$

To prove this result fill follow the steps outlined below:

i) Without loss of generality, one may assume that $\sup g < \infty$ and furthermore assume $4aT < 1$. Consider the function

$$v(x,t) = u(x,t) - \mu w(x,t) \quad x \in (-\infty,\infty) \quad t \in [0,T]$$

where

$$w(x,t) = \frac{1}{(T+\epsilon-t)^{\frac{1}{2}}} \exp\left(\frac{|x|^2}{T+\epsilon-t}\right)$$

What initial value problem does $v(x,t)$ satisfy? How do the initial values of $v(x,t)$ compare to the initial values of $u(x,t)$, i.e. what is the relation between $v(x,0)$ and $\sup_y g(y)$

ii) Using the growth condition for $u(x,t)$,show that there exists a sufficiently large R such that

(5) $$v(x,t) \leq \sup_{y\in(-\infty,\infty)} g(y) \quad |x| \geq R, t \in [0,T]$$

iii) Apply the maximum principle for $v(x,t)$ on the finite domain $|x| \leq R, t \in [0,T]$ to conclude that

$$v(x,t) \leq \sup_{y\in(-\infty,\infty)} g(y) \quad x \in (-\infty,\infty), t \in [0,T]$$

iv) The above result was valid for all values of μ. Take the limit $\mu \to 0$ to conclude that u satisfies the maximum principle

v) Use the maximum principle to show that the heat equation coupled with the growth conditions on u has a unique solution.