PRACTICE PROBLEM SET 1

Review questions:

1. Differentiating under the integral:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,s)\,ds = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}(x,s) + b'(x)\, f(x, b(x)) - a'(x)\, f(x, a(x))$$

Compute

a)

$$\frac{d}{dx}\int_0^{-x} e^{x+s} ds$$

b)

$$\frac{d}{dx}\int_x^{\pi} \left(\sin(x)\, y + y^2 x\right) dy$$

2. Chain rule in multivariable case:

Consider the chain $(s,t) \to (x,y) \to u$. If u is a function of x, y and if x, y are differentiable functions of s, t then

$$u_s = u_x \frac{\partial x}{\partial s} + u_y \frac{\partial y}{\partial s}$$

$$u_t = u_x \frac{\partial x}{\partial t} + u_y \frac{\partial y}{\partial t}$$

Consider the function of two variables

$$u(x,y) = \cos(x)\sin(y) + \sin(x)\cos(y) + x^2 - 2xy + y^2$$

Consider the change of variables $\xi = x + y$ and $\eta = x - y$. Compute $u_\xi, u_\eta, u_{\xi\xi}, u_{\eta\eta}$ and $u_{\xi\eta}$.

3. Series and radius of convergence

a. Compute the radii of convergence for the series of functions:

$$\sum_{n=1}^{\infty} \frac{x^{2n+1}}{n}, \sum_{n=1}^{\infty} \frac{x^n}{n!}, \sum_{n=1}^{\infty} \frac{x^{2n}}{3^n}$$

b. Compute the derivative of the following series of functions, you may express your answer as another series

$$\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n}, \sum_{n=1}^{\infty} x^{n!}$$

4. Parametrization of curves

a. Describe what curves each of these parametrizations represent

$$\left(t, t^2\right), \left(t, \frac{1}{t}\right), \left(\sin(\pi t), \cos(\pi t)\right), \left(\sqrt{1-t^2}, t\right), (t, 2t, 3t) \quad t \in (0,1)$$

Note, that the last one is in three dimensions.

b. Describe what the following parametrization represent in three dimensions

$$\left(R\cos(\theta), R\sin(\theta), z\right) \quad R = 1, \theta \in [0, 2\pi), z \in [0,1]$$

$$\left(2\cos(\theta)\sin(\phi), \cos(\theta)\cos(\phi), 3\sin(\theta)\right), \quad \theta \in [0,\pi], \phi \in (0, 2\pi]$$

5. Green's theorem and divergence theorem

i) Let D be a bounded domain in $\mathbb{R}^2$ with piecewise C^1 boundary ∂D. Let ∂D be parametrized so that the boundary is traversed once with D on the left. Let $p(x,y), q(x,y)$ be C^1 functions then

$$\int\int_D (q_x - p_y)\, dxdy = \int_{\partial D} pdx + qdy$$

ii) Let D be a bounded spatial domain with piecewise C^1 boundary ∂D. Let $\boldsymbol{n}$ be the unit outward normal on ∂D. Let $\boldsymbol{f}(\boldsymbol{x})$ be a C^1 vector field on D. Then

$$\int\int\int_D \nabla \cdot \boldsymbol{f}(\boldsymbol{x}) = \int\int_{\partial D} \boldsymbol{f} \cdot \boldsymbol{n} dS$$

a. Evaluate

$$\int\int_{\partial D} (2x, 3y, 4xy) \cdot \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}} dS$$

where D is the sphere $x^2 + y^2 + z^2 = 16$

b. Compute the integral

$$\int_{\partial D} y^3 dx - x^3 dy$$

where D is an annulus with inner radius 1 and outer radius 2.