1) (6 pts) True or false. Provide an explanation for your answer as well.
   i) The differential operator \( L[u] = -u'' \) defined on the interval \( x \in [0, 1] \) with periodic boundary conditions, i.e. \( u(0) = u(1), \ u'(0) = u'(1) \) has no negative eigenvalues.

   **Solution:**
   True, with periodic boundary conditions \( (u, -u'') = \int_0^1 (u')^2 \, dx \geq 0 \).
   
   ii) Consider the following solution to the heat equation defined on the domain \( t \in [0, 1], \ x \in [0, \pi] \)
   \[ u(x, t) = e^{-4t} \sin(2x) - 3e^{-t} \sin(x) + 17e^{-25t} \sin(5x). \]

   The function \( u(x, t) \) achieves its maximum at \( (x, t) = (0, \frac{\pi}{2}) \) and its minimum at \( (x, t) = \left( \frac{1}{3}, \frac{\pi}{3} \right) \).

   **Solution:**
   False, since \( u \) satisfies the heat equation, the maximum and minimum have to be achieved on the boundary.

   iii) Laplace’s equation on the interval with Dirichlet data \( f, g \) (both constants)
   \[ u''(x) = 0, \quad 0 < x < 1, \]
   \[ u(0) = f, \]
   \[ u(1) = g, \]
   is well posed.

   **Solution:**
   Yes, the solution is given by \( u = f + (g - f) x \), which is unique and depends continuously on \( f \) and \( g \).

---

2) (8 pts) Consider the solution of the 1D wave equation
   \[ u_{tt} = u_{xx}, \quad x \in \mathbb{R}, t > 0 \]
   \[ u(x, 0) = \phi(x), \]
   \[ u_t(x, 0) = \psi(x), \]
   where \( \phi(x) = \psi(x) = 0 \) for all \( |x| \geq 1 \). Prove that the energy
   \[ E(t) = KE(t) + PE(t) = \int_{\mathbb{R}} \left( \frac{1}{2} u_t^2 + \frac{1}{2} u_x^2 \right) \, dx, \]
   is conserved, i.e. \( E(t) \) is a constant. Here \( KE(t) \) is the kinetic energy and \( PE(t) \) is the potential energy. Furthermore, show that there exists \( T_0 \) such that \( KE(t) = PE(t) \) for all \( t > T_0 \) (Hint: Compute the kinetic and potential energy in terms of \( \phi \) and \( \psi \)).

   **Solution:**
   \[ u(x, t) = \frac{1}{2} (\phi(x + t) + \phi(x - t)) + \frac{1}{2} \int_{x-t}^{x+t} \psi(s) \, ds \]
   \[ \partial_x u = \frac{1}{2} (\phi'(x + t) + \phi'(x - t)), \]
   \[ \partial_t u = \frac{1}{2} (\phi'(x + t) + \phi'(x - t) + \psi(x + t) + \psi(x - t)) = (a - b + c + d) \]
   \[ (\partial_x u)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac - 2ad + 2bc - 2bd + 2cd \]
   \[ (\partial_t u)^2 = a^2 + b^2 + c^2 + d^2 - 2ab + 2ac + 2ad - 2bc - 2bd + 2cd \]
   Clearly the only terms that differ in the expression are of the form \( f(x + t) \, g(x - t) \), where \( f = \phi' \) or \( \psi \) and \( g = \phi' \) or \( \psi \).
   For \( t > 2 \), for each \( x \) either \( |x + t| > 1 \) or \( |x - t| > 1 \). Thus, the product of \( f(x + t) \, g(x - t) \) is pointwise 0 for \( t > 2 \).
3) (10 pts) Compute all separation of variables solutions of
\[ u_t = u_{xx}, \quad 0 < x < 1, 0 < t \]
\[ u(0, t) = u_x(0, t) \]
\[ u(1, t) = u_x(1, t) \]
\[ u(x, 0) = \phi(x). \]

You may assume that the operator
\[ \mathcal{L}[X] = -X'' \]
with the boundary conditions
\[ X(0) = X'(0), \quad X(1) = X'(1), \]
is hermitian and has only positive eigenvalues. (+1 Bonus point for proving this result). Suppose \( X_n(x) \) are the eigenfunctions for the operator above, and suppose
\[ \phi(x) = \sum_{n=1}^{\infty} A_n X_n(x), \]
where \(|A_n| \leq M\). Write down a formula for \( A_n \). Compute the solution \( \phi(x) \) corresponding to \( \phi(x) \) defined above.

(1 Bonus point question) In what mode of convergence does the series representation of \( u(x, t_0) \) converge for \( t_0 > 0 \)?

Solution:
The eigen functions are
\[ X_n(x) = A_n (\cos(n \pi x) + n \pi \sin(n \pi x)) \]

Hermitian follows from the fact that
\[ (u, Lv) - (Lu, v) = uv' - vu'|_0^1 = 0 \]

Positivity follows from the fact that
\[ (u, Lu) = (u', u') \geq 0 \]

\[ u(x, t) = \sum_{n=0}^{\infty} A_n (\cos(n \pi x) + n \pi \sin(n \pi x)) e^{-(n \pi)^2 t} \]

The convergence is uniform for \( t = t_0 > 0 \) since \( \sum_n e^{-(n \pi)^2 t} < \infty \).

4) (6 pts) Solve the following PDEs:
i) \[ 2u_x + 3u_y = 0, \]
\[ u(x, 0) = f(x), \]
Compute \( u(x, y) \) and sketch the characteristics.

Solution:
\[ u(x, y) = f \left( x - \frac{2}{3} y \right) \]

ii) Compute the solution to the PDE
\[ u_t = u_{xx}, \quad 0 < x < \infty, t > 0, \]
\[ u(0, t) = t^2, \quad t > 0, \]
\[ u(x, 0) = 0, \quad 0 < x < \infty. \]

Solution:
Let
\[ f_{\text{odd}}(x, t) = \begin{cases} 2t & x > 0 \\ -2t & x < 0 \end{cases} \]

The solution is given by
\[ u(x, t) = t^2 + \int_0^t \int_{\mathbb{R}} S(x - y, t - s) f_{\text{odd}}(y, s) \, dy \, ds \]