MIDTERM - MATH 247

- Total points: 30

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1) (6 pts) True or false. Provide an explanation for your answer as well.

i) The differential operator $\mathcal{L}[u] = -u''$ defined on the interval $x \in [0,1]$ with periodic boundary conditions, i.e. $u(0) = u(1)$, $u'(0) = u'(1)$ has no negative eigenvalues.

Solution:

True, with periodic boundary conditions $(u, -u'') = \int_0^1 (u')^2 \, dx \geq 0$.

ii) Consider the following solution to the heat equation defined on the domain $t \in [0,1]$, $x \in [0,\pi]$

$$u(x,t) = e^{-4t}\sin(2x) - 3e^{-t}\sin(x) + 17e^{-25t}\sin(5x) \ .$$

The function $u(x,t)$ achieves it's maximum at $(x,t) = \left(0, \frac{\pi}{2}\right)$ and it's minimum at $(x,t) = \left(\frac{1}{e}, \frac{\pi}{3}\right)$.

Solution:

False, since u satisfies the heat equation, the maximum and minimum have to be achieved on the boundary

iii) Laplace's equation on the interval with Dirichlet data f, g (both constants)

$$\begin{aligned} u''(x) &= 0\,, \quad 0 < x < 1\,, \\ u(0) &= f\,, \\ u(1) &= g\,, \end{aligned}$$

is well posed.

Solution:

Yes, the solution is given by $u = f + (g-f)x$, which is unique and depends continuously on f and g.

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2) (8 pts) Consider the solution of the 1D wave equation

$$\begin{aligned} u_{tt} &= u_{xx}\,, \quad x \in \mathbb{R}\,, t > 0 \\ u(x,0) &= \phi(x)\ , \\ u_t(x,0) &= \psi(x)\ , \end{aligned}$$

where $\phi(x) = \psi(x) = 0$ for all $|x| \geq 1$. Prove that the energy

$$E(t) = KE(t) + PE(t) = \int_{\mathbb{R}} \left(\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2\right) dx\,,$$

is conserved, i.e. $E(t)$ is a constant. Here $KE(t)$ is the kinetic energy and $PE(t)$ is the potential energy. Furthermore, show that there exists T_0 such that $KE(t) = PE(t)$ for all $t > T_0$ (Hint: Compute the kinetic and potential energy in terms of ϕ and ψ) .

Solution:

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\phi(x+t) + \phi(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} \psi(s)\,ds \\ \partial_x u &= \frac{1}{2}\left(\phi'(x+t) + \phi'(x-t) + \psi(x+t) - \psi(x-t)\right) = a + b + c - d \\ \partial_t u &= \frac{1}{2}\left(\phi'(x+t) - \phi'(x-t) + \psi(x+t) + \psi(x-t)\right) = (a - b + c + d) \\ (\partial_x u)^2 &= a^2 + b^2 + c^2 + d^2 + 2ab + 2ac - 2ad + 2bc - 2bd - 2cd \\ (\partial_t u)^2 &= a^2 + b^2 + c^2 + d^2 - 2ab + 2ac + 2ad - 2bc - 2bd + 2cd \end{aligned}$$

Clearly the only terms that differ in the expression are of the form $f(x+t)\,g(x-t)$, where $f = \phi'$ or ψ and $g = \phi'$ or ψ. For $t > 2$, for each x either $|x+t| > 1$ or $|x-t| > 1$. Thus, the product of $f(x+t)\,g(x-t)$ is pointwise 0 for $t > 2$.

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3) (10 pts) Compute all separation of variables solutions of

$$\begin{aligned} u_t &= u_{xx}\,, \quad 0<x<1\,,0<t \\ u(0,t) &= u_x(0,t) \\ u(1,t) &= u_x(1,t) \\ u(x,0) &= \phi(x)\ . \end{aligned}$$

You may assume that the operator

$$\mathcal{L}[X] = -X''$$

with the boundary conditions

$$X(0) = X'(0)\ , \quad X(1) = X'(1)\ ,$$

is hermitian and has only positive eigenvalues. (+1 Bonus point for proving this result). Suppose $X_n(x)$ are the eigenfunctions for the operator above, and suppose

$$\phi(x) = \sum_{n=1}^{\infty} A_n X_n(x)\ ,$$

where $|A_n| \leq M$. Write down a formula for A_n. Compute the solution $\phi(x)$ corresponding to $\phi(x)$ defined above.

(+1 Bonus point question) In what mode of convergence does the series representation of $u(x,t_0)$ converge for $t_0 > 0$?

Solution:

The eigen functions are

$$X_n(x) = A_n(\cos(n\pi x) + n\pi\sin(n\pi x))$$

Hermitian follows from the fact that

$$(u, Lv) - (Lu, v) = uv' - vu'|_0^1 = 0$$

Positivity follows from the fact that

$$(u, Lu) = (u', u') \geq 0$$

$$u(x,t) = \sum_{n=0}^{\infty} A_n(\cos(n\pi x) + n\pi\sin(n\pi x))\, e^{-(n\pi)^2 t}$$

The convergence is uniform for $t = t_0 > 0$ since $\sum_n e^{-(n\pi)^2 t} < \infty$.

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4) (6 pts) Solve the following PDEs:

i)

$$\begin{aligned} 2u_x + 3u_y &= 0\,, \\ u(x,0) &= f(x)\ , \end{aligned}$$

Compute $u(x,y)$ and sketch the characteristics.

Solution:

$$u(x,y) = f\left(x - \frac{2}{3}y\right)$$

ii) Compute the solution to the PDE

$$\begin{aligned} u_t &= u_{xx}\,, \quad 0<x<\infty\,, t>0\,, \\ u(0,t) &= t^2\,, \quad t>0\,, \\ u(x,0) &= 0\,, \quad 0<x<\infty\,. \end{aligned}$$

Solution:

Let

$$f_{\text{odd}}(x,t) = \begin{cases} 2t & x>0 \\ -2t & x<0 \end{cases}$$

The solution is given by

$$u(x,t) = t^2 + \int_0^t \int_{\mathbb{R}} S(x-y, t-s)\, f_{\text{odd}}(y,s)\, dy\, ds$$