PROBLEM SET 6

DUE DATE: - APR 25

- **Chap 7, 9.1**
- Questions are either directly from the text or a small variation of a problem in the text.
- Collaboration is okay, but final submission must be written individually. Mention all collaborators on your submission.
- The terms in the bracket indicate the problem number from the text.

Section 7.1

1) (Prob 5, Pg 184) Prove Dirichlet's Principle for Neumann boundary condition. It asserts that among all real-valued functions $w(\boldsymbol{x})$ on D, the quantity

$$E[w] = \frac{1}{2}\iiint_D |\nabla w|^2 \, , d\boldsymbol{x} - \iint_{\partial D} hw \, dS \, ,$$

is the smallest for $w = u$, where u is the solution of the Neumann problem

$$-\Delta u = 0 \quad \text{in } D \quad \frac{\partial u}{\partial n} = h(\boldsymbol{x}) \quad \text{on } \partial D \, ,$$

where h satisfies the constraint

$$\iint_{\partial D} h(\boldsymbol{x}) \, dS = 0 \, .$$

Note that there are no restrictions on w as opposed to the Dirichlet principle for Dirichlet boundary conditions, the function $h(\boldsymbol{x})$ appears in the energy and the energy does not change if you add a constant to w. Comment on the last bit in context of solutions to the Neumann problem for Laplace's equation.

Solution:

Let w be the minimizer of the energy above, then given any function $v \in C^2$, consider

$$f_v(\epsilon) = E[w + \epsilon v] = \frac{1}{2}\iiint_D |\nabla(w + \epsilon v)|^2 \, d\boldsymbol{x} - \iint_{\partial D} h(w + \epsilon v) \, dS \, .$$

Since w is a minimizer, $f'(0) = 0$. Using our standard calculation

$$f'(\epsilon) = \iiint_D \nabla v \cdot \nabla w \, d\boldsymbol{x} + \epsilon \iiint_D |\nabla v|^2 - \iint_{\partial D} hv \, dS$$

$$f'(0) = \iiint_D \nabla v \cdot \nabla w \, d\boldsymbol{x} - \iint_{\partial D} hv \, dS$$

On using integration by parts for the first term, we get

$$0 = f'(0) = \iiint_D v \Delta w \, d\boldsymbol{x} - \iint_{\partial D} v \left(h - \frac{\partial w}{\partial n}\right) dS \, .$$

The above identity holds for all functions v and hence we must have

$$\Delta w = 0 \quad \boldsymbol{x} \in D$$
$$\frac{\partial w}{\partial n} = h \quad \boldsymbol{x} \in \partial D$$

Now suppose u satisfies the above Neumnn boundary value problem. Then

$$
\begin{aligned}
E[u] &= E[w+(u-w)] \\
&= E[w] + E[u-w] + \iiint_D \nabla w \cdot \nabla (u-w)\ d\boldsymbol{x} \\
&= E[w] + \frac{1}{2}\iiint_D |\nabla (u-w)|^2\, d\boldsymbol{x} - \iint_{\partial D} (u-w)\, h\, dS + \iiint_D \nabla w \cdot \nabla (u-w)\ d\boldsymbol{x} \\
&= E[w] + \frac{1}{2}\iiint_D |\nabla (u-w)|^2\, d\boldsymbol{x} + \iiint_D (u-w)\, \Delta w\, d\boldsymbol{x} \\
&= E[w] + \frac{1}{2}\iiint_D |\nabla (u-w)|^2\, d\boldsymbol{x}\,.
\end{aligned}
$$

Thus, $E[u] \geq E[w]$.

The fact that the energy does not change by adding constants is consistent with the fact that we can only compute solutions of the Neumann equation up to a constant.

Section 7.2

2) (Prob 1, Pg 187) Derive the representation formula for hamronic functions in two dimensions

$$
u(\boldsymbol{x}_0) = \frac{1}{2\pi}\int_{\partial D}\left[u(\boldsymbol{x})\frac{\partial}{\partial n}\left(\log|\boldsymbol{x}-\boldsymbol{x}_0|\right) - \frac{\partial u}{\partial n}\log|\boldsymbol{x}-\boldsymbol{x}_0|\right] ds
$$

Solution:

Without loss of generality, let $\boldsymbol{x}_0 = 0$. Let $v(\boldsymbol{x}) = -\frac{1}{2\pi}\log|\boldsymbol{x}|$ and apply Green's second identity to the domain $D \setminus B_\epsilon(0)$. Then

$$
0 = \iint_{D\setminus B_\epsilon(0)} (u\Delta v - v\Delta u)\, d\boldsymbol{x} = \int_{\partial D\setminus B_\epsilon(0)} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS\,.
$$

The boundary again separates into two parts, ∂D and $\partial B_\epsilon(0)$ where the normal to $\partial B_\epsilon(0)$ is inward facing.

$$
\begin{aligned}
\int_{\partial D\setminus B_\epsilon(0)} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS = {} & \frac{1}{2\pi}\int_{\partial D}\left(-u(\boldsymbol{x})\frac{\partial}{\partial n}\log|\boldsymbol{x}| + \frac{\partial u}{\partial n}\log|\boldsymbol{x}|\right) dS + \\
& \frac{1}{2\pi}\int_{\partial B_\epsilon(0)}\left(-u(\boldsymbol{x})\frac{\partial}{\partial n}\log|\boldsymbol{x}| + \frac{\partial u}{\partial n}\log|\boldsymbol{x}|\right) dS\,.
\end{aligned}
\tag{1}
$$

On $\partial B_\epsilon(0)$, $\boldsymbol{n} = -\hat{r}$ and $\boldsymbol{x} = \epsilon\hat{r}$, and $dS = \epsilon d\theta$ where $\hat{r}$ is the unit vector in the outward radial direction.

$$
\begin{aligned}
-\frac{1}{2\pi}\int_{\partial B_\epsilon(0)} u(\boldsymbol{x})\frac{\partial}{\partial n}\log|\boldsymbol{x}|\ dS &= -\frac{1}{2\pi}\int_0^{2\pi} u(\boldsymbol{x})\left(\frac{\boldsymbol{x}\cdot\boldsymbol{n}}{|\boldsymbol{x}|^2}\right)\epsilon d\theta \\
&= \frac{1}{2\pi}\int_0^{2\pi} u(\epsilon,\theta)\, d\theta \\
\lim_{\epsilon\to 0} -\frac{1}{2\pi}\int_{\partial B_\epsilon(0)} u(\boldsymbol{x})\frac{\partial}{\partial n}\log|\boldsymbol{x}|\ dS &= u(0)\ ,
\end{aligned}
$$

where the last equality follows from the continuity of u.

Similarly,

$$\frac{1}{2\pi}\int_{\partial B_\epsilon(0)}\frac{\partial u}{\partial n}\log|\boldsymbol{x}|\,dS = \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial u}{\partial n}\log|\boldsymbol{x}|\,\epsilon d\theta$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\frac{\partial u}{\partial n}\log(\epsilon)\,\epsilon d\theta$$
$$\left|\frac{1}{2\pi}\int_{\partial B_\epsilon(0)}\frac{\partial u}{\partial n}\log|\boldsymbol{x}|\,dS\right| = \left|\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial u}{\partial n}\log(\epsilon)\,\epsilon d\theta\right|$$
$$\leq \left|\frac{\partial u}{\partial n}\right|_\infty \log(\epsilon)\,\epsilon$$
$$\lim_{\epsilon\to 0}\left|\frac{1}{2\pi}\int_{\partial B_\epsilon(0)}\frac{\partial u}{\partial n}\log|\boldsymbol{x}|\,dS\right| = 0$$

Taking the limit $\epsilon \to 0$ in Equation1, we get

$$\frac{1}{2\pi}\int_{\partial D}\left(-u(\boldsymbol{x})\frac{\partial}{\partial n}\log|\boldsymbol{x}| + \frac{\partial u}{\partial n}\log|\boldsymbol{x}|\right)dS + u(0) = 0\,,$$

which proves the result.

Section 7.3

3) (Prob 1, Pg 190) Show that the Green's function is unique. (Hint: Take the difference of two of them)

Solution:

Let $G_1(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} + H_1(\boldsymbol{x},\boldsymbol{y})$ and $G_2(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} + H_2(\boldsymbol{x},\boldsymbol{y})$ be two Green's functions. Then, their difference $G_1 - G_2 = H_1(\boldsymbol{x},\boldsymbol{y}) - H_2(\boldsymbol{x},\boldsymbol{y})$ is a harmonic in the $\boldsymbol{x}$ variable for every $\boldsymbol{y}$ and moreover, for a fixed $\boldsymbol{y}$, $H_1 - H_2 = 0$ on the boundary since $G_1 = G_2 = 0$ on the boundary as a function of $\boldsymbol{x}$. Thus, for a fixed $\boldsymbol{y}$, the difference of two Green's functions is a harmonic function in $\boldsymbol{x}$ and 0 on the boundary. By uniqueness of the interior Dirichlet problem, we conclude that the difference must be identically zero in the interior.

Section 7.4

4) (Prob 7,8 Pg 196) a) If $u(x,y) = f\left(\frac{x}{y}\right)$ is a harmonic function, solve the ODE satisfied by f.

b) Show that $\partial_r u \equiv 0$, where $r = \sqrt{x^2+y^2}$

c) Suppose $v(x,y)$ is any $\{y>0\}$ such that $\partial_r v \equiv 0$. Show that $v(x,y)$ is a function of the quotient $\frac{x}{y}$.

d) Find the boundary values $\lim_{y\to 0} u(x,y) = h(x)$

e) Find the harmonic function in the half plane $\{y>0\}$ with boundary data $h(x)=1$ for $x>0$ and $h(x)=0$ for $x<0$.

f) Find the harmonic function in the half plane $\{y>0\}$ with boundary data $h(x)=1$ for $x>a$ and $h(x)=0$ for $x<a$.

Solution:

a)

$$f(s) = A\arctan(s) + B$$

b) $x = r\cos(\theta)$, $y = r\sin(\theta)$

$$\partial_r u = \frac{d}{dx}f\left(\frac{x}{y}\right)\cdot\frac{\partial x}{\partial r} + \frac{d}{dy}f\left(\frac{x}{y}\right)\cdot\frac{\partial y}{\partial r}$$
$$= f'\left(\frac{x}{y}\right)\cdot\frac{1}{y}\cdot\frac{x}{r} - f'\left(\frac{x}{y}\right)\cdot\frac{x}{y^2}\cdot\frac{y}{r}$$
$$= 0$$

c)

$$\partial_r v(x,y) = \frac{x}{r}\partial_x v + \frac{y}{r}\partial_y v = 0$$

Use method of characteristics to conclude that

$$v = f\left(\frac{x}{y}\right).$$

d)

$$h(x) = \begin{cases}\frac{1}{2}\pi A + B & x>0\\ -\frac{1}{2}\pi A + B & x<0\end{cases}$$

e)

$$u(x,y)=\frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{x}{y}\right)$$

f)

$$u(x,y)=\frac{1}{2}+\frac{1}{\pi}\arctan\left(\frac{x-a}{y}\right)$$

5) (Prob 17, Pg 197) a) Find the Green's function for the quadrant

$$Q=\{(x,y):\,x>0\,,y>0\}\,.$$

b) Use the answer in part (a) to solve the Dirichlet problem

$$\begin{aligned}\Delta u&=0 \quad \text{in } Q\\ u(0,y)&=g(y) \quad y>0\\ u(x,0)&=h(x) \quad x>0.\end{aligned}$$

Solution:
Use method of images to place appropriate charges in each quadrant.
a)

$$\begin{aligned}G(\boldsymbol{x},\boldsymbol{x}_0)=&-\frac{1}{2\pi}\log\left(\sqrt{(x-x_0)^2+(y-y_0)^2}\right)-\frac{1}{2\pi}\log\left(\sqrt{(x+x_0)^2+(y+y_0)^2}\right)\\&+\frac{1}{2\pi}\log\left(\sqrt{(x-x_0)^2+(y+y_0)^2}\right)+\frac{1}{2\pi}\log\left(\sqrt{(x+x_0)^2+(y-y_0)^2}\right)\end{aligned}$$

b)

$$\begin{aligned}u(x,y)=&\int_0^\infty xg(\eta)\left[\frac{1}{(y-\eta)^2+x^2}-\frac{1}{(y+\eta)^2+x^2}\right]\frac{d\eta}{\pi}+\\&\int_0^\infty yh(\xi)\left[\frac{1}{(x-\xi)^2+y^2}-\frac{1}{(x+\xi)^2+y^2}\right]\frac{d\xi}{\pi}\,.\end{aligned}$$

6) (Prob 21, Pg 198) The Neumann function $N(\boldsymbol{x},\boldsymbol{y})$ for a domain D is defined exactly like the Green's function with the following conditions:

$$N(\boldsymbol{x},\boldsymbol{y})=-\frac{1}{4\pi|x-y|}+H(\boldsymbol{x},\boldsymbol{y})$$

where $H(\boldsymbol{x},\boldsymbol{y})$ is a harmonic function of $\boldsymbol{x}$ for each fixed $\boldsymbol{y}$, and

$$\frac{\partial N}{\partial n}=c \quad \boldsymbol{x}\in\partial D$$

for a suitable constant c.
a) Show that $c=\frac{1}{A}$ where A is the area of the boundary ∂D.
b) State and prove the analog of Theorem 7.3.1, expressing the solution of the Neumann problem in terms of the Neumann function.
Solution:
a) Without loss of generality, we may assume that $\boldsymbol{y}=0$.

$$\begin{aligned}0=\iint_{D\setminus B_\epsilon(0)}\Delta N(\boldsymbol{x},\boldsymbol{y})\,dV&=\int_{\partial D\setminus B_\epsilon(0)}\frac{\partial N}{\partial n}dS\\&=\int_{\partial D}\frac{\partial N}{\partial n}\,dS+\int_{\partial B_\epsilon(0)}\frac{\partial N}{\partial n}\,dS\\&=cA-1\,.\end{aligned}$$

Thus, $c=\frac{1}{A}$.
b)

$$u(\boldsymbol{x})=-\int_{\partial D}N(\boldsymbol{x},\boldsymbol{y})\frac{\partial u}{\partial n}(\boldsymbol{y})\,dS+\frac{1}{A}\int_{\partial D}u\,dS\,.$$

7) (Prob 22: Pg 198) Solve the Neumann problem in the half plane:

$$\Delta u=0 \quad \text{in } \{y>0\}\,,\frac{\partial u}{\partial y}(x,0)=h(x)$$

and $u(x,y)$ is bounded at ∞.

Solution:

$$u(x,y) = C + \int_{-\infty}^{\infty} h(x-\xi)\log\left(y^2+\xi^2\right) d\xi\,,$$

boundedness follows from the fact that $\int_{-\infty}^{\infty} h(x-\xi)\, d\xi = 0\,.$

Section 9.1

8) (Prob 1, Pg 233) Find all three-dimensional plane waves: i.e., all the solutions of the wave equation of the form $u(\boldsymbol{x},t) = f(\boldsymbol{k}\cdot\boldsymbol{x} - ct)$ where $\boldsymbol{k}$ is a fixed vector and f is a function of one variable

Solution:

Either $|\boldsymbol{k}| = 1$ and any arbitary f works or

$$u = a + b(\boldsymbol{k}\cdot\boldsymbol{x} - ct)$$

9) (Prob 8, Pg 234) Consider the equation

$$\partial_{tt}u - c^2\Delta u + m^2 u = 0\,,$$

where $m > 0$, known as the Klein-Gordon equation.

a) What is the energy? Show that it is a constant.

b) Prove the causality principle for this equation.

Solution:

a)

$$E(t) = \frac{1}{2}\iint_{\mathbb{R}^2}\left((u_t)^2 + c^2|\nabla u|^2 + m^2u^2\right) d\boldsymbol{x}$$

$$\begin{aligned}
\frac{d}{dt}E(t) &= \iint_{\mathbb{R}^2}\left(u_tu_{tt} + c^2\nabla u\cdot\nabla u_t + m^2uu_t\right) d\boldsymbol{x} \\
&= \iint_{\mathbb{R}^2}\left(u_tu_{tt} - c^2u_t\Delta u + m^2uu_t\right) d\boldsymbol{x} \quad \text{(integration by parts)} \\
&= 0
\end{aligned}$$

b) Proof of causality follows in exactly the same fashion, since

$$\partial_t\left(\frac{1}{2}(u_t)^2 + \frac{1}{2}c^2|\nabla u|^2 + \frac{1}{2}m^2u^2\right) - c^2\nabla\cdot(u_t\nabla u) = \left(u_{tt} - c^2\Delta u + m^2u\right)u_t$$