PROBLEM SET 5

DUE DATE: - APR 11

- **Chap 5**
- Questions are either directly from the text or a small variation of a problem in the text.
- Collaboration is okay, but final submission must be written individually. Mention all collaborators on your submission.
- The terms in the bracket indicate the problem number from the text.

Section 6.1
1) (Prob 7, Pg 160) Solve $u_{xx} + u_{yy} + u_{zz} = 1$ in the spherical shell $1 < r < 2$, with $u(1, \theta, \phi) = u(2, \theta, \phi) = 0$ for all θ, ϕ.
Solution:
We set

$$u = u_p + u_h \,,$$

where u_p satisfies the Poisson equation, and u_h fixes the boundary condition. It is easier to take a radially symmetric solution to the inhomogeneous problem.

$$u_p(r, \theta, \phi) = \frac{r^2}{6} \,.$$

It is easy to check that

$$\Delta u_p = 1 \,.$$

Then on imposing the boundary condition on u, we get the following boundary condition for u_h.

$$u_h(1, \theta, \phi) = -\frac{1}{6} \,, \quad u_h(2, \theta, \phi) = -\frac{4}{6} \,.$$

The only two radially symmetric solutions are $\frac{1}{r}$ and 1. Since both the boundary data is radially symmetric, the solution must be a linear combination of both of these solutions:

$$u_h(r, \theta, \phi) = \frac{c_1}{r} + c_2 \,.$$

Solving for c_1, c_2 we get

$$u_h(r) = \frac{1}{r} - \frac{7}{6} \,.$$

The total solution is given by

$$u(r, \theta, \phi) = \frac{r^2}{6} + \frac{1}{r} - \frac{7}{6} \,.$$

2) (Prob 13, Pg 160) A function u is subharmonic in D if it satisfies $\Delta u \geq 0$ in D. Prove that it's maximum value is attained on the boundary. Note that the same is not true for the minimum value.
Solution:
This problem can be solved via either of the two routes that we've used to prove maximum principle. A way to prove it is to use $v = u + \epsilon |r|^2$ and take the limit as $\epsilon \to 0$. But we will proceed the alternate route using an alternate form of the mean value property.

$$\iint_{\partial D} \frac{\partial u}{\partial r} dS = \iiint_D \Delta u \, dV \geq 0 \,.$$

Proceeding as in the proof for the mean value theorem, we then conclude

$$\partial_r \left[\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(r, \theta, \phi) \sin(\theta) \, d\theta \, d\phi \right] = \iint_{\partial D} \frac{\partial u}{\partial r} dS \geq 0 \,.$$

Thus

$$u(\boldsymbol{x}) \leq \frac{1}{4\pi r^2} \iint_{\partial B_r(\boldsymbol{x})} u(\boldsymbol{y}) \, dS_{\boldsymbol{y}} \quad \forall r \geq 0 \,.$$

Again, by the same reasoning as before, we can use the above result to conclude that

$$u(\boldsymbol{x}) \leq \frac{1}{\frac{4}{3}\pi r^3} \iiint_{B_r(\boldsymbol{x})} u(\boldsymbol{y})\, dV \quad \forall r \geq 0$$

Now, we proceed with the clopen argument exactly in the text to show that maximum principle holds.

Let M be the maximum value of u in D. Let $A = u^{-1}\{M\}$. Then A is a closed subset of D. Furthermore suppose $\boldsymbol{x}_0 \in A$. Then applying the inequality version of the mean value theorem, we get

$$u(\boldsymbol{x}_0) \leq \frac{1}{\frac{4}{3}\pi r^3} \iiint_{B_r(\boldsymbol{x}_0)} u(\boldsymbol{y})\, dV\,.$$

Since $u(\boldsymbol{y}) \leq M$ and $u(\boldsymbol{x}_0) = M$, we conclude that the only way the above inequality can hold is if $u(\boldsymbol{y}) = M$ for all $\boldsymbol{y} \in B_r(\boldsymbol{x}_0)$, i.e. if $\boldsymbol{x}_0 \in A$, then $B_r(\boldsymbol{x}_0) \in A$, i.e. A is an open set. Since A is both open and closed, we conclude that $A = D$ or $A = \phi$.

Section 6.2

3) (Prob 1, Pg 164) Solve $u_{xx} + u_{yy} = 0$ in the rectangle $0 < x < 1$, $0 < y < 2$ with the following boundary conditions:

$$\begin{aligned} u_x &= -1 \quad x = 0 \\ u_y &= 2 \quad y = 0 \\ u_x &= 0 \quad x = 1 \\ u_y &= 0 \quad y = 2\,. \end{aligned}$$

Solution:

Solution strategy: In this case, we can take a shortcut

$$u(x,y) = v(x) + h(y)$$

where $v(x)$ satisfies the ode $v'' = 1$ with $v' = -1$ at $x = 0$ and $v' = 0$ at $x = 1$, the solution to which is given by

$$v(x) = \frac{1}{2}x^2 - x + c\,.$$

The reason we needed the one in there is to guarantee that the compatibility condition for the Neumann problem, i.e.

$$\int_0^1 v'' = v'(1) - v'(0)$$

is satisfied.

Then h has to satisfy the ODE,

$$h'' = -1$$

with $h'(0) = 2$ and $h'(2) = 0$. It is easy to see that the compatibility condition for this bvp is automatically satisfied in this case (you could conclude this from the fact that $u(x,y)$ satisfies the compatibility condition too).

Thus,

$$h(y) = -\frac{1}{2}y^2 + 2y + c\,.$$

Combining, both of these, we get

$$u(x,y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 - x + 2y + c$$

4) (Prob 7, Pg 165) Find the harmonic function in the semi-infinite strip $\{0 \leq x \leq \pi\,, 0 \leq y < \infty\}$ that satisfy the boundary conditions:

$$u(0,y) = u(\pi,y) = 0\,, \quad u(x,0) = h(x)\,, \quad \lim_{y\to\infty} u(x,y) = 0\,.$$

b) What would be the issue if the condition at ∞ is not imposed?

Solution:

The separation of variables solutions in this case are given by

$$u(x,y) = \sin(nx)\, e^{ny} \quad \text{and} \quad \sin(nx)\, e^{-ny}\,.$$

In order to obtain solutions which satisfy the boundary condition at ∞, we have to discard the solutions at grow exponentially as $y \to \infty$. Thus, we represent our solution as a linear combination of

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin(nx) e^{-ny} .$$

On imposing the boundary conditions at $y = 0$ and if the sine series of $h(x)$ is given by

$$h(x) = \sum_{n=1}^{\infty} A_n \sin(nx) \ ,$$

then the solution u is given by

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(nx) e^{-ny} .$$

b) If we do not impose the decay conditions at ∞ then

$$u(x,y) = \sum_{n=1}^{\infty} \left(a_n e^{-ny} + b_n e^{ny}\right) \sin(nx) \ .$$

For the h given above, any set of values $\{a_n\}$ and $\{b_n\}$ which satisfy

$$a_n + b_n = A_n$$

will be a solution to the PDE. So we have non-uniqueness.

Section 6.3

5) (Prob 2, Pg 172) Solve $u_{xx} + u_{yy} = 0$ in the disk $\{r < a\}$ with the boundary condition

$$u(a,\theta) = 1 + 3\sin(\theta) \ .$$

Solution: The bounded solutions separation of variables in the interior of a disk are given by

$$u(r,\theta) = r^n \cos(n\theta) \quad \text{and} \quad u(r,\theta) = r^n \sin(n\theta) \quad n > 0$$

and $u(r,\theta) = 1$ for $n = 1$. On imposing the boundary condition for $r = a$ and using the orthogonality of the basis, we conclude that the solution is given by

$$u(r,\theta) = 1 + 3\left(\frac{r}{a}\right)\sin(\theta)$$

Section 6.4

6) (Prob 1, Pg 175) Solve $u_{xx} + u_{yy} = 0$ in the exterior $\{r > a\}$ of the disk, with the boundary condition $u(a,\theta) = 1 + 3\sin(\theta)$ and the condition that u remains bounded as $r \to \infty$.

Solution:

The bounded solutions separation of variables in the exterior of a disk are given by

$$u(r,\theta) = r^{-n} \cos(n\theta) \quad \text{and} \quad u(r,\theta) = r^{-n} \sin(n\theta) \quad n > 0$$

and $u(r,\theta) = 1$ for $n = 1$. On imposing the boundary condition for $r = a$ and using the orthogonality of the basis, we conclude that the solution is given by

$$u(r,\theta) = 1 + 3\left(\frac{a}{r}\right)\sin(\theta)$$

7) (Prob 4, Pg 176) Derive Poisson's formula for the exterior of a circle.

Solution:

The separation of variables solutions in this case are given by

$$u(r,\theta) = r^{-n} \sin(n\theta) \quad \text{and} \quad r^{-n} \cos(n\theta) \quad n > 0$$

and $u(r,\theta) = 1$ for $n = 0$. Then

$$u(r,\theta) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n r^{-n} \sin(n\theta) + b_n r^{-n} \cos(n\theta) \ .$$

On imposing the boundary conditions and expressing a_n and b_n as integrals of the boundary data h, we get

$$a_n = \frac{a^n}{\pi} \int_0^{2\pi} h(\phi) \cos(n\phi)\, d\phi$$
$$b_n = \frac{a^n}{\pi} \int_0^{2\pi} h(\phi) \sin(n\phi)\, d\phi$$

Plugging it back into the expression for u, we get

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} h(\phi)\, d\phi + \int_0^{2\pi} h(\phi) \left(\sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \cos(n(\theta - \phi)) \right) d\phi$$

The above geometric series converges absolutely for all $r > a$, so all changes in order of integration and summation are valid. Explicitly computing the above sum as in class, we get

$$u(r,\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{r^2 - a^2}{a^2 - 2ar\cos(\theta - \phi) + r^2} h(\phi)\, d\phi$$

The only thing different in the above formula is the change in sign from the poisson formula for the interior of the disk.