PROBLEM SET 3

DUE DATE: - MAR 9

- **Sections 4.1 - 4.3**
- Questions are either directly from the text or a small variation of a problem in the text.
- Collaboration is okay, but final submission must be written individually. Mention all collaborators on your submission.
- The terms in the bracket indicate the problem number from the text.

Section 4.1

1) (Prob 4,5 Pg 89) Consider waves in a resistant medium which satisfy the following PDE:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} - r u_t \quad 0 < x < \ell \\ u(0,t) &= u(\ell,t) = 0 \quad \forall t > 0 \\ u(x,0) &= \phi(x) \quad 0 < x < \ell \\ \partial_t u(x,0) &= \psi(x) \quad 0 < x < \ell \,, \end{aligned}$$

where r is a constant. Write down a series expansion for the following cases:

i)

$$0 < r < \frac{2\pi c}{\ell}$$

ii)

$$\frac{2\pi c}{\ell} < r < \frac{4\pi c}{\ell} \,.$$

You may assume that the initial conditions can be represented using an appropriate Fourier series.

Section 4.2

2) (Prob 2, Pg 92) Solve the wave equation with mixed boundary conditions using separation of variables, i.e. write down a series representation for the solution. You may assume that the initial conditions can be represented using an appropriate Fourier series.:

$$\begin{aligned} u_{tt} &= k u_{xx} \quad 0 < x < \ell \\ u_x(0,t) &= u(\ell,t) = 0 \\ u(x,0) &= \phi(x) \quad 0 < x < \ell \\ \partial_t u(x,0) &= \psi(x) \quad 0 < x < \ell \end{aligned}$$

3) (Prob 3, Pg 92) Solve the Schrodinger equation $u_t = iku_{xx}$ for real k in the interval $0 < x < \ell$ with mixed boundary conditions $u_x(0,t) = u(\ell,t) = 0$.

4) (Prob 4, Pg 92) (Periodic boundary conditions) Consider diffusion inside an enclosed circular tube. Let it's length be 2ℓ. Let x denote the arclength parameter. The concentration of the diffusing substance satisfies

$$\begin{aligned} u_t &= k u_{xx} \quad -\ell \le x \le \ell \\ u(-\ell,t) &= u(\ell,t) \\ \partial_x u(-\ell,t) &= \partial_x u(\ell,t) \,, \end{aligned}$$

Show that the solution is given by

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi x}{\ell}\right) + B_n \sin\left(\frac{n\pi x}{\ell}\right) \right) \exp\left(-\frac{n^2\pi^2 kt}{\ell^2}\right) \,.$$

Section 4.3

5) (Prob 2, Pg 100) Consider the Robin eigenvalue value problem

$$X'' = -\lambda X$$
$$X'(0) - a_0 X(0) = X'(\ell) + a_\ell X(\ell) = 0\,.$$

a) Show that $\lambda = 0$ is an eigenvalue if and only if $a_0 + a_\ell = -a_0 a_\ell \ell$.
b) Find the eigenfunctions corresponding to the zero eigenvalue.

6) (Prob 4, Pg 100) Consider the Robin eigenvalue problem. If $a_0 < 0$, $a_\ell < 0$ and $-a_0 - a_\ell < a_0 a_\ell \ell$, show that there are two negative eigenvalues. (Hint: Show that the rational curve

$$y = -\frac{(a_0 + a_\ell)\,\gamma}{\gamma^2 + a_0 a_\ell}\,,$$

has a single maximum and crosses the line $y = 1$ in two places. Deduce that it crosses the tanh curve in two places as well.

7) (Prob 18, Pg 102-103). A tuning fork may be regarded as a pair of vibrating flexible bars with a certain degree of stiffness. The governing equation for such a fork is given by the fourth order PDE

$$u_{tt} + c^2 u_{xxxx} = 0 \quad 0 < x < \ell$$
$$u(0,t) = u_x(0,t) = 0 \quad \text{(Fixed end/clamped boundary conditions)}$$
$$u_{xx}(\ell,t) = u_{xxx}(\ell,t) = 0 \quad \text{(Free end/No stress at the end)}\ .$$

a) Separate the time and space variables to get the eigenvalue problem

$$X'''' = \lambda X\,.$$

b) Show that 0 is not an eigenvalue.
c) Assuming that all the eigenvalues are positive, write them as $\lambda = \beta^4$ and find the equation for β.
d) Find the frequencies of vibration.
e) Compare the answer in part (d) with the overtones of the vibrating string by comparing at the ratio β_2^2/β_1^2 . Explain why you hear an almost pure tone when you listen to a tuning fork.