Section 2.4
1) (Prob 9, Pg 53) Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x, 0) = x^2$ by the following method.

i) Show that $u_{xxx}$ also satisfies the diffusion equation with zero initial condition.

**Solution:**

\[
\begin{align*}
& \frac{\partial}{\partial x} u = ku_{xx} \\
& \frac{\partial}{\partial x} (u_t) = k\frac{\partial}{\partial x} u_{xx} \\
& \frac{\partial}{\partial x} (u_{xxx}) = k\frac{\partial}{\partial x} (u_{xx}) \quad \text{(Since partials commute)}
\end{align*}
\]

ii) Conclude that $u_{xxx}(x, t) \equiv 0$ for all $(x, t)$ assuming $|u(x, t)| \leq Me^{\alpha x^2}$ for all $x, t$. (Refer to practice problem set 2)

**Solution:** $u_{xxx}(x, 0) \equiv 0$ since $u(x, 0) = x^2$. Since under the growth conditions, we have uniqueness for the heat equation on the real line, we conclude that $u_{xxx}(x, t) = 0$

iii) Using ii, show that $u(x, t) = A(t)x^2 + B(t)x + C(t)$

**Solution:** Follows from integrating the above expression 3 times.

iv) Plug in the above expression for $u(x, t)$ in the differential equation, to obtain a system of differential equations for $A(t), B(t), C(t)$. (Hint: the functions $1, x, x^2$ are linearly independent)

\[
A'(t)x^2 + B'(t)x + C'(t) = \frac{\partial}{\partial t} u = ku_{xx} = 2kA(t)
\]

\[
\begin{align*}
A'(t) &= 0 \\
B'(t) &= 0 \\
C'(t) &= 2kA(t)
\end{align*}
\]

Solving the above system, we get

\[
\begin{align*}
A(t) &= a \\
B(t) &= b \\
C(t) &= 2kat + c
\end{align*}
\]

v) Using the initial conditions, obtain initial values for $A(0), B(0)$ and $C(0)$ and solve the above system of differential equations to compute $u(x, t)$

\[
A(0)x^2 + B(0)x + C(0) = x^2
\]

\[
A(0) = 1, B(0) = 0, C(0) = 0.
\]

\[
u(x, t) = x^2 + 2kt
\]
2) (Prob 15, Pg 53) Using the energy method, prove uniqueness of the diffusion problem with Neumann boundary conditions:

\[
\begin{align*}
    u_t - ku_{xx} &= f(x,t) & 0 < x < 1, t > 0 \\
    u(x,0) &= \phi(x) \\
    u_x(0,t) &= g(t) \\
    u_x(0,1) &= h(t)
\end{align*}
\]

**Solution:** To prove uniqueness for the neumann problem, we consider the problem above with \( f(x,t) \equiv 0, \phi(x) \equiv 0, g(t) \equiv 0 \) and \( h(t) \equiv 0 \) and show that the solution \( u(x,t) \equiv 0 \).

\[
e(t) = \int_0^1 u^2(x,t) \, dx
\]

\[
e'(t) = \int_0^1 2uu_t \, dx
\]

\[
= 2k \int_0^1 uu_{xx} \, dx
\]

\[
= 2k u_x [1]_0^1 - 2k \int_0^1 u_x^2 \, dx \quad \text{(Integration by parts)}
\]

\[
= -2k \int_0^1 u_x^2 \, dx \leq 0 \quad (u_x(0,t) = u_x(1,t) = 0)
\]

Thus, the energy is a decreasing function of time. Moreover \( e(0) = 0 \) since \( u(x,0) \equiv 0 \). \( e(t) \) is a non negative, decreasing function of time, which is \( 0 \) at \( t = 0 \), so we conclude that \( e(t) \equiv 0 \) and thus \( u(x,t) \equiv 0 \).

3) (Prob 16, Pg 54) Solve the diffusion equation with constant dissipation:

\[
\begin{align*}
    u_t - ku_{xx} + bu &= 0 & -\infty < x < \infty \\
    u(x,0) &= \phi(x)
\end{align*}
\]

where \( b > 0 \) is a constant by setting \( u(x,t) = e^{-bt}v(x,t) \).

**Solution:**

\[
v(x,t) = e^{bt}u(x,t)
\]

\[
\partial_t v(x,t) = e^{bt} \partial_t u(x,t) + be^{bt} u(x,t)
\]

\[
\partial_{xx} v(x,t) = e^{bt} \partial_{xx} u(x,t)
\]

\[
\partial_t v - k\partial_{xx} v = e^{bt} (\partial_t u + bu - k\partial_{xx} u) = 0
\]

Thus, \( v(x,t) \) satisfies the heat equation with initial data \( v(x,0) = e^{bt}\phi(x) = \phi(x) \). The solution for \( v(x,t) \) is given by

\[
v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y) \, dy
\]

\[
u(x,t) = e^{-bt} \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y) \, dy
\]

Section 3.1

4) (Prob 2, Pg 60) Solve \( u_t = ku_{xx}; u(x,0) = 0; u(0,t) = 1 \) on the half life \( 0 < x < \infty \). (Hint: Set \( v(x,t) = u(x,t) - 1 \).)

**Solution:** Let \( v(x,t) = u(x,t) - 1 \). Then \( v \) satisfies the differential equation

\[
\partial_t v = k\partial_{xx} v
\]

\[
v(0,t) = 0 \quad \forall t > 0
\]

\[
v(x,0) = -1 \quad 0 < x < \infty
\]
The solution is given by

\[ v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left( \exp \left( -\frac{(x-y)^2}{4kt} \right) - \exp \left( -\frac{(x+y)^2}{4kt} \right) \right) \phi(y) \, dy \]

\[ = -\frac{1}{\sqrt{4\pi kt}} \int_{0}^{\infty} \left( \exp \left( -\frac{(x-y)^2}{4kt} \right) - \exp \left( -\frac{(x+y)^2}{4kt} \right) \right) \, dy \]

\[ = -\text{erf} \left( \frac{x}{\sqrt{4kt}} \right) \]

\[ u(x,t) = 1 - \text{erf} \left( \frac{x}{\sqrt{4kt}} \right) \]

---

**Section 3.2**

5) (Prob 3, Pg 66) A wave \( f(x+ct) \) travels along a semi-infinite string \( 0 < x < \infty \) for \( t > 0 \). Find the solution \( u(x,t) \) of the string for \( t > 0 \) if the end \( x = 0 \) is fixed, i.e. \( u(0,t) = 0 \). Plot the solution in the three different regimes. Repeat the same exercise if \( u_x(0,t) = 0 \). Comment on the results.

**Solution:**

**Dirichlet case**

\[ u(x,t) = \begin{cases} 
  f(x+ct) & x > ct \\
  f(x+ct) - f(ct-x) & x < ct 
\end{cases} \]

**Neumann case**

\[ u(x,t) = \begin{cases} 
  f(x+ct) & x > ct \\
  f(x+ct) + f(ct-x) & x < ct 
\end{cases} \]

---

6) (Prob 5, Pg 66) Solve \( u_{tt} = 4u_{xx} \) for \( 0 < x < \infty \), \( u(0,t) = 0 \), \( u(x,0) = 1 \) and \( u_t(x,0) = 0 \) using the reflection method. Find the location of the singularity of the solution in the \((x,t)\) space.

**Solution:**

\[ u(x,t) = \begin{cases} 
  1 & x \geq 2t \\
  0 & x < 2t 
\end{cases} \]

The singularity is on the line \( x = 2t \).

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**Section 3.3**

7) (Prob 2, Pg 71) Solve the completely inhomogeneous diffusion problem on the half line

\[ v_t - kv_{xx} = f(x,t) \quad 0 < x < \infty, \quad 0 < t < \infty \]

\[ v(0,t) = h(t), \quad v(x,0) = \phi(x), \]

by setting \( V(x,t) = v(x,t) - h(t) \).

**Solution:** Let \( V(x,t) = v(x,t) - h(t) \). Then \( V(x,t) \) satisfies

\[ V_t - kV_{xx} = v_t - h'(t) - kv_{xx} = f(x,t) - h'(t) \]

\[ V(0,t) = 0 \]

\[ V(x,0) = \phi(x) - h(0) \]
By Duhamel’s principle, the solution to the above inhomogeneous problem is given by
\[ V(x, t) = \int_{0}^{t} \int_{-\infty}^{\infty} \left[ S(x - y, t - s) - S(x + y, t - s) \right] \cdot \left[ f(y, s) - h'(s) \right] \, dy \, ds + \]
\[ \int_{-\infty}^{\infty} \left[ S(x - y, t) - S(x + y, t) \right] \cdot [\phi(y) - h(0)] \, dy \]
\[ v(x, t) = h(t) + V(x, t) \]

Section 3.4
8) (Prob 12.13, Pg 80) Derive the solution of the fully inhomogeneous wave equation on the half-line
\[ v_{tt} - c^2 v_{xx} = f(x, t), \quad 0 < x < \infty \]
\[ v(x, 0) = \phi(x) \quad v_t(x, 0) = \psi(x) \]
by setting \( V(x, t) = v(x, t) - h(t) \). Find the solution for \( h(t) = t^2, \phi(x) = x \) and \( \psi(x) = 0 \).

Solution: Let \( V(x, t) = v(x, t) - h(t) \). Then \( V(x, t) \) satisfies the differential equation
\[ V_{tt} - c^2 V_{xx} = f(x, t) - h''(t) \]
\[ V(x, 0) = \phi(x) - h(0) \]
\[ V_t(x, 0) = \psi(x) - h'(0) \]
\[ V(0, t) = 0. \]

Let
\[ \overline{\phi}_{\text{odd}}(x) = \begin{cases} \phi(x) - h(0) & x \geq 0 \\ -\phi(-x) - h(0) & x < 0 \end{cases} \]
\[ \overline{\psi}_{\text{odd}}(x) = \begin{cases} \psi(x) - h'(0) & x \geq 0 \\ -\psi(-x) - h'(0) & x < 0 \end{cases} \]
\[ \overline{f}_{\text{odd}}(x, t) = \begin{cases} f(x, t) - h''(t) & x \geq 0 \ t > 0 \\ -f(-x, t) - h''(t) & x < 0 \ t > 0 \end{cases} \]

By Duhamel’s principle, the solution is given by
\[ V(x, t) = \frac{1}{2} \left[ \overline{\phi}_{\text{odd}}(x + ct) + \overline{\phi}_{\text{odd}}(x - ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \overline{\psi}_{\text{odd}}(y) \, dy + \frac{1}{2c} \int_{x-ct}^{x+ct} \overline{f}_{\text{odd}}(y, s) \, dy \, ds \]
\[ v(x, t) = V(x, t) + h(t) \]
\[ v(x, t) = \begin{cases} x & x \geq ct \\ x + (t - \frac{c}{t})^2 & 0 \leq x \leq ct \end{cases} \]

9) (Stability to small perturbations for the heat equation) Consider the inhomogeneous heat equation on the real line
\[ u_t - ku_{xx} = f(x, t), \quad -\infty < x < \infty, \quad t > 0 \]
\[ u(x, 0) = \phi(x). \]

Show that the solutions are stable under small perturbations, i.e. Show that
\[ \|u\|_{L^\infty(\mathbb{R} \times [0, T])} = \sup_{x \in \mathbb{R}, t \in [0, T]} |u(x, t)| \leq T \|f\|_{L^\infty(\mathbb{R} \times [0, T])} + \|\phi\|_{L^\infty(\mathbb{R})}. \]

Use the formula for the solution and the fact that
\[ \int_{-\infty}^{\infty} S(x, t) \, dx = 1, \forall t > 0. \]

Comment on why the above result implies stability of solutions to the data for the heat equation.

Solution:
\[ u(x, t) = \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) f(y, s) \, dy \, ds + \int_{-\infty}^{\infty} S(x - y, t) \phi(y) \, dy \]

\[ |u(x, t)| = \left| \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) f(y, s) \, dy \, ds + \int_{-\infty}^{\infty} S(x - y, t) \phi(y) \, dy \right| \]

\[ \leq \left| \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) f(y, s) \, dy \, ds \right| + \left| \int_{-\infty}^{\infty} S(x - y, t) \phi(y) \, dy \right| \quad \text{(Triangle ineq)} \]

\[ \leq \int_0^t \int_{-\infty}^{\infty} |S(x - y, t - s) f(y, s)| \, dy \, ds + \int_{-\infty}^{\infty} |S(x - y, t) \phi(y)| \, dy \quad \left( \int f \leq \int |f| \right) \]

\[ \leq \|f\|_{L^\infty(\mathbb{R} \times [0, T])} \int_0^t \int_{-\infty}^{\infty} S(x - y, t - s) \, dy \, ds + \|\phi\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{\infty} S(x - y, t) \phi(y) \, dy \quad (S > 0) \]

\[ \leq \|f\|_{L^\infty(\mathbb{R} \times [0, T])} \int_0^t ds + \|\phi\|_{L^\infty(\mathbb{R})} \quad \left( \int S \, dx = 1 \right) \]

\[ \leq T \|f\|_{L^\infty(\mathbb{R} \times [0, T])} + \|\phi\|_{L^\infty(\mathbb{R})} \]

The above result implies stability because small perturbations in \( f \) and \( \phi \), result in small perturbations in the solution \( u \).