PROBLEM SET 2

DUE DATE: FEB 28

- **Sections 2.4-3.5**
- Questions are either directly from the text or a small variation of a problem in the text.
- Collaboration is okay, but final submission must be written individually. Mention all collaborators on your submission.
- The terms in the bracket indicate the problem number from the text.

Section 2.4

1) (Prob 9, Pg 53) Solve the diffusion equation $u_t = ku_{xx}$ with the initial condition $u(x,0) = x^2$ by the following method.

i) Show that u_{xxx} also satisfies the diffusion equation with zero initial condition.

Solution:

$$\begin{aligned} u_t &= ku_{xx} \\ \partial_{xxx}(u_t) &= k\partial_{xxx}u_{xx} \\ \partial_t(u_{xxx}) &= k\partial_{xx}(u_{xxx}) \quad \text{(Since partials commute)} \end{aligned}$$

ii) Conclude that $u_{xxx}(x,t) \equiv 0$ for all (x,t) assuming $|u(x,t)| \le Me^{ax^2}$ for all x, t. (Refer to practice problem set 2)

Solution: $u_{xxx}(x,0) \equiv 0$ since $u(x,0) = x^2$. Since under the growth conditions, we have uniqueness for the heat equation on the real line, we conclude that $u_{xxx}(x,t) = 0$

iii) Using ii, show that

$$u(x,t) = A(t)x^2 + B(t)x + C(t)$$

Solution: Follows from integrating the above expression 3 times.

iv) Plug in the above expression for $u(x,t)$ in the differential equation, to obtain a system of differential equations for $A(t), B(t), C(t)$. (Hint: the functions $1, x, x^2$ are linearly independent)

$$A'(t)x^2 + B'(t)x + C'(t) = \partial_t u = ku_{xx} = 2kA(t)$$

$$\begin{aligned} A'(t) &= 0 \\ B'(t) &= 0 \\ C'(t) &= 2kA(t) \end{aligned}$$

Solving the above system, we get

$$\begin{aligned} A(t) &= a \\ B(t) &= b \\ C(t) &= 2kat + c \end{aligned}$$

v) Using the initial conditions, obtain initial values for $A(0), B(0)$ and $C(0)$ and solve the above system of differential equations to compute $u(x,t)$

$$A(0)x^2 + B(0)x + C(0) = x^2$$

$$A(0) = 1,\ B(0) = 0,\ C(0) = 0.$$

$$u(x,t) = x^2 + 2kt$$

---

2) (Prob 15, Pg 53) Using the energy method, prove uniqueness of the diffusion problem with Neumann boundary conditions:

$$\begin{aligned} u_t - ku_{xx} &= f(x,t) \quad 0 < x < 1\,, t > 0 \\ u(x,0) &= \phi(x) \\ u_x(0,t) &= g(t) \\ u_x(0,1) &= h(t) \end{aligned}$$

Solution: To prove uniqueness for the neumann problem, we consider the problem above with $f(x,t) \equiv 0$, $\phi(x) \equiv 0$, $g(t) \equiv 0$ and $h(t) \equiv 0$ and show that the solution

$$u(x,t) \equiv 0\,.$$

$$\begin{aligned} e(t) &= \int_0^1 u^2(x,t)\,dx \\ e'(t) &= \int_0^1 2uu_t\,dx \\ &= 2k\int_0^1 uu_{xx}\,dx \\ &= 2kuu_x|_0^1 - 2k\int_0^1 u_x^2\,dx \quad \text{(Integration by parts)} \\ &= -2k\int_0^1 u_x^2\,dx \le 0 \quad (u_x(0,t) = u_x(1,t) = 0) \end{aligned}$$

Thus, the energy is a decreasing function of time. Moreover $e(0) = 0$ since $u(x,0) \equiv 0$. $e(t)$ is a non negative, decreasing function of time, which is 0 at $t = 0$, so we conclude that $e(t) \equiv 0$ and thus $u(x,t) \equiv 0$.

---

3) (Prob 16, Pg 54) Solve the diffusion equation with constant dissipation:

$$\begin{aligned} u_t - ku_{xx} + bu &= 0 \quad -\infty < x < \infty \\ u(x,0) &= \phi(x)\,, \end{aligned}$$

where $b > 0$ is a constant by setting $u(x,t) = e^{-bt}v(x,t)$.

Solution:

$$\begin{aligned} v(x,t) &= e^{bt}u(x,t) \\ \partial_t v(x,t) &= e^{bt}\partial_t u(x,t) + be^{bt}u(x,t) \\ \partial_{xx}v(x,t) &= e^{bt}\partial_{xx}u(x,t) \end{aligned}$$

$$\partial_t v - k\partial_{xx}v = e^{bt}(\partial_t u + bu - k\partial_{xx}u) = 0$$

Thus, $v(x,t)$ satisfies the heat equation with initial data $v(x,0) = e^{b\cdot 0}\phi(x) = \phi(x)$. The solution for $v(x,t)$ is given by

$$\begin{aligned} v(x,t) &= \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\phi(y)\,dy \\ u(x,t) &= e^{-bt}\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}}\phi(y)\,dy \end{aligned}$$

Section 3.1

4) (Prob 2, Pg 60) Solve $u_t = ku_{xx}$; $u(x,0) = 0$; $u(0,t) = 1$ on the half life $0 < x < \infty$. (Hint: Set $v(x,t) = u(x,t) - 1$.)

Solution: Let $v(x,t) = u(x,t) - 1$. Then v satisfies the differential equation

$$\begin{aligned} \partial_t v &= k\partial_{xx}v \\ v(0,t) &= 0 \quad \forall t > 0 \\ v(x,0) &= -1 \quad 0 < x < \infty \end{aligned}$$

The solution is given by

$$\begin{aligned} v(x,t) &= \frac{1}{\sqrt{4\pi kt}}\int_0^\infty \left(\exp\left(-\frac{(x-y)^2}{4kt}\right) - \exp\left(-\frac{(x+y)^2}{4kt}\right)\right)\phi(y)\, dy \\ &= -\frac{1}{\sqrt{4\pi kt}}\int_0^\infty \left(\exp\left(-\frac{(x-y)^2}{4kt}\right) - \exp\left(-\frac{(x+y)^2}{4kt}\right)\right) dy \\ &= -\mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right) \\ u(x,t) &= 1 - \mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right) \end{aligned}$$

Section 3.2

5) (Prob 3, Pg 66) A wave $f(x+ct)$ travels along a semi-infinite string $0 < x < \infty$ for $t > 0$. Find the solution $u(x,t)$ of the string for $t > 0$ if the end $x = 0$ is fixed, i.e. $u(0,t) = 0$. Plot the solution in the three different regimes. Repeat the same exercise if $u_x(0,t) = 0$. Comment on the results.

Solution:

Dirichlet case

$$u(x,t) = \begin{cases} f(x+ct) & x > ct \\ f(x+ct) - f(ct-x) & x < ct \end{cases}$$

Neumann case

$$u(x,t) = \begin{cases} f(x+ct) & x > ct \\ f(x+ct) + f(ct-x) & x < ct \end{cases}$$

---

6) (Prob 5, Pg 66) Solve $u_{tt} = 4u_{xx}$ for $0 < x < \infty$, $u(0,t) = 0$, $u(x,0) = 1$ and $u_t(x,0) = 0$ using the reflection method. Find the location of the singularity of the solution in the (x,t) space.

Solution:

$$u(x,t) = \begin{cases} 1 & x \geq 2t \\ 0 & x < 2t \end{cases}$$

The singularity is on the line $x = 2t$.

Section 3.3

7) (Prob 2, Pg 71) Solve the completely inhomogeneous diffusion problem on the half line

$$\begin{aligned} v_t - kv_{xx} = f(x,t) \quad 0 < x < \infty\,, \quad 0 < t < \infty \\ v(0,t) = h(t)\,, \quad v(x,0) = \phi(x)\,, \end{aligned}$$

by setting $V(x,t) = v(x,t) - h(t)$.

Solution: Let $V(x,t) = v(x,t) - h(t)$. Then $V(x,t)$ satisfies

$$\begin{aligned} V_t - kV_{xx} &= v_t - h'(t) - kv_{xx} = f(x,t) - h'(t) \\ V(0,t) &= 0 \\ V(x,0) &= \phi(x) - h(0) \end{aligned}$$

By Duhamel's principle, the solution to the above inhomogeneous problem is given by

$$\begin{aligned} V(x,t) &= \int_0^t \int_{-\infty}^{\infty} \left[S(x-y,t-s) - S(x+y,t-s)\right] \cdot \left[f(y,s) - h'(s)\right] \, dy \, ds + \\ &\int_{-\infty}^{\infty} \left[S(x-y,t) - S(x+y,t)\right] \cdot \left[\phi(y) - h(0)\right] \, dy \\ v(x,t) &= h(t) + V(x,t) \end{aligned}$$

Section 3.4

8) (Prob 12,13, Pg 80) Derive the solution of the fully inhomogeneous wave equation on the half-line

$$\begin{aligned} v_{tt} - c^2 v_{xx} &= f(x,t) \,, \quad 0 < x < \infty \\ v(x,0) &= \phi(x) \quad v_t(x,0) = \psi(x) \\ v(0,t) &= h(t) \end{aligned}$$

by setting $V(x,t) = v(x,t) - h(t)$. Find the solution for $h(t) = t^2$, $\phi(x) = x$ and $\psi(x) = 0$.

Solution: Let $V(x,t) = v(x,t) - h(t)$. Then $V(x,t)$ satisfies the differential equation

$$\begin{aligned} V_{tt} - c^2 V_{xx} &= f(x,t) - h''(t) \\ V(x,0) &= \phi(x) - h(0) \\ V_t(x,0) &= \psi(x) - h'(0) \\ V(0,t) &= 0 \,. \end{aligned}$$

Let

$$\overline{\phi}_{odd}(x) = \begin{cases} \phi(x) - h(0) & x \geq 0 \\ -(\phi(-x) - h(0)) & x < 0 \end{cases}$$

$$\overline{\psi}_{odd}(x) = \begin{cases} \psi(x) - h'(0) & x \geq 0 \\ -(\psi(-x) - h'(0)) & x < 0 \end{cases}$$

$$\overline{f}_{odd}(x,t) = \begin{cases} f(x,t) - h''(t) & x \geq 0 \, t > 0 \\ -(f(-x,t) - h''(t)) & x < 0 \, t > 0 \end{cases}$$

By Duhamel's principle, the solution is given by

$$V(x,t) = \frac{1}{2}\left[\overline{\phi}_{odd}(x+ct) + \overline{\phi}_{odd}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \overline{\psi}_{odd}(y) \, dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \overline{f}_{odd}(y,s) \, dy \, ds$$

$$v(x,t) = V(x,t) + h(t)$$

$$v(x,t) = \begin{cases} x & x \geq ct \\ x + \left(t - \frac{x}{c}\right)^2 & 0 \leq x \leq ct \end{cases}$$

---

9) (Stability to small perturbations for the heat equation) Consider the inhomogeneous heat equation on the real line

$$\begin{aligned} u_t - k u_{xx} &= f(x,t) \quad -\infty < x < \infty \,, \quad t > 0 \\ u(x,0) &= \phi(x) \,. \end{aligned}$$

Show that the solutions are stable under small perturbations, i.e. Show that

$$\|u\|_{\mathbb{L}^\infty(\mathbb{R}\times[0,T])} = \sup_{x\in\mathbb{R}, t\in[0,T]} |u(x,t)| \leq T\|f\|_{\mathbb{L}^\infty(\mathbb{R}\times[0,T])} + \|\phi\|_{\mathbb{L}^\infty(\mathbb{R})}.$$

Use the formula for the solution and the fact that

$$\int_{-\infty}^{\infty} S(x,t) \, dx = 1 \quad \forall t > 0 \,.$$

Comment on why the above result implies stability of solutions to the data for the heat equation.

Solution:

$$
\begin{aligned}
u(x,t) &= \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\, f(y,s)\ dy\, ds + \int_{-\infty}^{\infty} S(x-y,t)\, \phi(y)\ dy \\
|u(x,t)| &= \left| \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\, f(y,s)\ dy\, ds + \int_{-\infty}^{\infty} S(x-y,t)\, \phi(y)\ dy \right| \\
&\le \left| \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\, f(y,s)\ dy\, ds \right| + \left| \int_{-\infty}^{\infty} S(x-y,t)\, \phi(y)\ dy \right| \quad \text{(Triangle ineq)} \\
&\le \int_0^t \int_{-\infty}^{\infty} |S(x-y,t-s)\, f(y,s)|\ dy\, ds + \int_{-\infty}^{\infty} |S(x-y,t)\, \phi(y)|\ dy \quad \left(\left| \int f \right| \le \int |f| \right) \\
&\le \|f\|_{\mathbb{L}^\infty(\mathbb{R}\times[0,T])} \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)\ dy\, ds + \|\phi\|_{\mathbb{L}^\infty(\mathbb{R})} \int_{-\infty}^{\infty} S(x-y,t)\, \phi(y)\ dy \quad (S>0) \\
&\le \|f\|_{\mathbb{L}^\infty(\mathbb{R}\times[0,T])} \int_0^t ds + \|\phi\|_{\mathbb{L}^\infty(\mathbb{R})} \quad \left(\int_{\mathbb{R}} S\, dx = 1 \right) \\
&\le T\|f\|_{\mathbb{L}^\infty(\mathbb{R}\times[0,T])} + \|\phi\|_{\mathbb{L}^\infty(\mathbb{R})}
\end{aligned}
$$

The above result implies stability because small perturbations in f and ϕ, result in small perturbations in the solution u.