PROBLEM SET 1

DUE DATE: FEB 14

- **Sections 1.2 - 2.3**
- Questions are either directly from the text or a small variation of a problem in the text.
- Collaboration is okay, but final submission must be written individually. Mention all collaborators on your submission.
- The terms in the bracket indicate the problem number from the text.

Section 1.2

1) (Prob 3,6, Pg 10) Solve the following equations and sketch some of the characteristics for each case.

a) $(1+x)\,u_x + u_y = 0$

b) $\sqrt{1-x^2}u_x + u_y = 0$

2) (Prob 11, Pg 10) Solve $au_x + bu_y = f(x,y)$ where $f(x,y)$ is aa given function and a, b are constants with $a \neq 0$. Express the solution in the form

$$u(x,y) = \frac{1}{\sqrt{a^2+b^2}} \int_L f\, ds + g(bx - ay)$$

where g is an arbitrary function of one variable, L is the characteristic line segment from the y axis to the point (x,y) and the integral is a line integral. (Hint: Use the coordinate method.)

Bonus: Where was the assumption $a \neq 0$ used in the above problem.

Section 1.3

3) (Prob 6, Pg 19) Consider the heat equation in a long cylinder where the temperature only depends on t and the distance r to the axis of the cylinder. Here $r = \sqrt{x^2+y^2}$ is the cylinder coordinate. From the three dimensional heat equation derive the equation

$$u_t = k\left(u_{rr} + \frac{u_r}{r}\right).$$

4) (Prob 8, Pg 19) For the hydrogen atom, let $e(t) = \int |u(t,\boldsymbol{x})|^2\, d\boldsymbol{x}$. Show that if $e(0) = 1$, then $e(t) = 1$ for all t. (Hint: compute $e'(t)$. Keep in mind that u is complex valued. Assume that $|u(t,\boldsymbol{x})| = 0$ for $|\boldsymbol{x}| > R(t)$ where $R(t) < \infty$.

5) (Prob 11, Pg 20) If $\nabla \times \boldsymbol{v} = \boldsymbol{0}$ in all of $\mathbb{R}^3$. Show that there exists a scalar function $\phi(x,y,z)$ such that $\boldsymbol{v} = \nabla\phi$.

Bonus: Is it true if $\nabla \times \boldsymbol{v} = \boldsymbol{0}$ on an arbirtrary domain D? Under what conditions on the domain D is it true?

Section 1.4

6) (Prob 6, Pg 25) Two homogeneous rods have the same cross section, specific heat c, and density ρ but different heat conductivities κ_1 and κ_2 and lengths L_1 and L_2. Let $k_j = \kappa_j/(c\rho)$ be their diffusion constants. They are welded together so that the temperature u and the flux κu_x are continuous. The left hand rod has its left end maintained at temperature 0. The right had rod has its right end at temperature T degrees.

a) Find the equilibrium temperature distribution in the composite rod.

b) Sketch it as a function of x in case $k_1 = 2$, $k_1 = 1$, $L_1 = 3$, $L_2 = 2$, $T = 10$.

Section 1.5

7) (Prob 1, Pg 27)Consider the boundary value ordinary differential equation

$$u''(x) + u(x) = 0\,, \quad u(0) = 0, u(L) = 0\,.$$

Clearly, the function $u(x) \equiv 0$ is a solution. Is the solution unique? Does the answer depend on L?

8) (Prob 4, Pg 28) Consider the Neumann problem

$$\Delta u = f(x,y,z) \quad \text{in D}$$
$$\frac{\partial u}{\partial \boldsymbol{n}} = 0 \quad \text{on } \partial D$$

a) Is the solution unique? What can we surely add to any solution to get another solution?
b) Use the divergence theorem and the PDE to show that

$$\int\int\int_D f(x,y,z)\, dx\, dy\, dz = 0$$

c) Give a physical interpretation of part a or part b either for heat flow or diffusion?

Section 2.1
9) (Prob 1, Pg 38) Solve $u_{tt} = 4u_{xx}$, $u(x,0) = e^x$, $u_t(x,0) = \sin(x)$.
10) (Prob 5, Pg 38) The hammer blow! A model for a note being played on a piano is the following.

$$u_{tt} = c^2 u_{xx} \quad u(x,0) = \phi(x) \quad u_t(x,0) = \psi(x)\ .$$

Let $\phi(x) \equiv 0$, and $\psi(x) = 1$ for $|x| \leq a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile $u(x)$ at each of the time $t = a/2c$, a/c, $3a/2c$, $2a/c$, $5a/c$.
11) (Prob 8, Pg 38) A spherical wave is a solution of the three-dimensional wave equation of the form $u(r,t)$, where r is the distance to the origin (the spherical coordinate). The wave equation takes the form

$$u_{tt} = c^2\left(u_{rr} + \frac{2}{r}u_r\right) \quad \text{("spherical wave equation")}$$

a) Change variables $v = ru$ to get the equation for $v : v_{tt} = c^2 v_{rr}$.
b) Solve for v given initial condition $u(r,0) = \phi(r)$ and $u_t(r,0) = \psi(r)$ where both $\phi(r)$ and $\psi(r)$ are even functions.
12) (Prob 9, Pg 38) Solve $u_{xx} - 3u_{xt} - 4u_{tt} = 0$, $u(x,0) = x^2$, $u_t(x,0) = e^x$. (Hint: Factor the operator)

Section 2.2
13) (Prob 5, Pg 41) Consider the damped string,

$$u_{tt} = c^2 u_{xx} - r u_t$$

Show that the energy decreases as a function of time. Prove uniqueness for the damped string.

Section 2.3
14) (Prob 1, Pg 45) Consider the solution $1 - x^2 - 2kt$ of the diffusion equation. Find the locations of its maximum and mimum in the closed rectangle $\{0 \leq x \leq 1, \quad 0 \leq t \leq T\}$.
15) (Prob 5, Pg 46) Consider the variable coefficient heat equation $u_t = x u_{xx}$
a) Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle $\{-2 \leq x \leq 2, 0 \leq t \leq 1\}$. Note that the maximum is not achieved on the boundary.
b) Where precisely does our proof of the maximum principle break down for this equation?