Lecture notes

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Preface

These are lecture notes for MATH 247: Partial differential equations with the sole purpose of providing reading material for topics covered in the lectures of the class. Several examples used here are reproduced from Partial Differential Equations, an Introduction: Walter Strauss.

Notation

Default notation unless mentioned.

- Independent variables: x, y, z (Spatial variables), t (time variable)
- Functions: u, f, g, h, ϕ, ψ
- $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$

1 Lec 1

1.1 What is a PDE?

Definition 1.1. *PDE Consider an unknonw function $u(x, y, \ldots)$ of several independent variables $(x, y, \ldots)$. A partial differential equation is a relation/identity/equation which relates the independent variables, the unknown function u and partial derivatives of the function u.*

Definition 1.2. *Order of a PDE The* order *of a PDE is the highest derivative that appears in the equation.*

The most general first order partial differential equation of two independent variables can be expressed as

$$F(x, y, u, u_x, u_y) = 0\,. \tag{1}$$

Similarly, the most general second order differential equation can be expressed as

$$F(x, y, u, u_x, u_y, u_{xx}, u_{yy}, u_{xy}) = 0\,. \tag{2}$$

Example 1.1. *Here are a few of examples of first order partial differential equations:*

$$u_t + u_x = 0\,, \quad \textit{Transport equation.} \tag{3}$$

$$u_t + uu_x = 0\,, \quad \textit{Burger's equation – model problem for shock waves} \tag{4}$$

$$u_t u_x = \sin(t + x)\,. \tag{5}$$

Example 1.2. *Here are a few examples of second order differential equations*

$$u_{xx} + u_{yy} + u_{zz} = 0 \quad \textit{Laplace's equation – Governing equations for electrostatics, potential flows, } \ldots\,. \tag{6}$$

$$u_t = u_{xx} + u_{yy}\,, \quad \textit{Heat equation/Diffusion equation}\,. \tag{7}$$

$$u_{tt} = u_{xx}\,, \quad \textit{Vibrating string.} \tag{8}$$

$$u_t = iu_{xx}\,, \quad \textit{Schrödinger's equation}\,. \tag{9}$$

$$u_t = u_{xx} + u^4\,, \quad \textit{Heat equation with radiation}\,. \tag{10}$$

$$xu_{xx} + yu_{yy} = (x + y)^2\,. \tag{11}$$

Remark 1.1. *The primary difference between a partial differential equation and an ordinary differential equation is that there is only one independent variable in an ordinary differential equation, . (the unknown function $u(t)$ is a function of the single independent variable t), where as in a partial differential equation, the unknown function is a function of more than one independent variable*

1.2 Properties of partial differential equations

In any partial differential equation, can alternatively be denoted in operator notation as

$$\mathcal{L}[u] = f(x, y \ldots)\,. \tag{12}$$

For example, the equation for a vibrating string (8) can be written in operator form as

$$\mathcal{L}_1[u] = u_{tt} - u_{xx}\,, \quad f_1(x,t) = 0\,. \tag{13}$$

Burger's equation (4) can be written in operator form as

$$\mathcal{L}_2[u] = u_t + uu_x\,, \quad f_2(x,t) = 0\,. \tag{14}$$

Equation (11) in operator form is given by

$$\mathcal{L}_3[u] = xu_{xx} + yu_{yy}\,, \quad f_3(x,y) = (x+y)^2\,. \tag{15}$$

1.2.1 Linearity

Definition 1.3. *Linearity of a PDE Given a partial differential equation in operator form*

$$\mathcal{L}[u] = f(x, y, \ldots)\,,$$

the equation is linear if the operator $\mathcal{L}$ is linear i.e. the operator $\mathcal{L}$ satisfies

$$\mathcal{L}[u+v] = \mathcal{L}[u] + \mathcal{L}[v]\,, \quad \mathcal{L}[cu] = c\mathcal{L}[u]\,, \tag{16}$$

for any function u, v and constant c. The operator $\mathcal{L}$ corresponding to a linear partial differential equation is called a linear operator.

Equations (8) and (11) are linear since the corresponding operators $\mathcal{L}_1$ and $\mathcal{L}_3$ satisfy the conditions for linearity:

$$\mathcal{L}_1[u+v] = (u+v)_{tt} - (u+v)_{xx} = u_{tt} - u_{xx} + v_{tt} - v_{xx} = \mathcal{L}_1[u] + \mathcal{L}_1[v]\,.$$

$$\mathcal{L}_1[cu] = (cu)_{tt} - (cu)_{xx} = cu_{tt} - cu_{xx} = c\mathcal{L}_1[u]\,.$$

$$\mathcal{L}_3[u+v] = x(u+v)_{xx} + y(u+v)_{yy} = xu_{xx} + yu_{yy} + xv_{xx} + yv_{yy} = \mathcal{L}_3[u] + \mathcal{L}_3[v]\,.$$

$$\mathcal{L}_3[cu] = x(cu)_{xx} + y(cu)_{yy} = cxu_{xx} + cyu_{yy} = c\mathcal{L}_3[u]\,.$$

However, equation (4) is non-linear, since

$$\mathcal{L}_2[u+v] = (u+v)_x + (u+v)(u+v)_y = u_x + (u+v)u_y + v_x + (u+v)v_y \neq u_x + uu_y + v_x + vv_y = \mathcal{L}_2[u] + \mathcal{L}_2[v]\,. \tag{17}$$

1.2.2 Homogeneity

Definition 1.4. *Homogeneous partial differential equation Given a linear partial differential equation*

$$\mathcal{L}[u] = f(x, y, \ldots)\,,$$

the equation is homogeneous if $f = 0$ and inhomogeneous otherwise.

Thus, the equation of a vibrating string (8) is linear since $f_1 = 0$ and equation (11) is inhomogeneous since $f_3 = (x+y)^2$

Exercise 1.1. *Classify equations 1–10 in the notes as linear or non-linear. If linear, futher classify them as homogeneous or inhomogeneous.*

1.2.3 Superposition principle

One important characteristic of linear differential equations is the superposition principle. Suppose $\mathcal{L}$ is a linear operator. Then if u_1 and u_2 are solutions to $\mathcal{L}[u] = 0$, then $v = c_1u_1 + c_2u_2$ also satisfies $\mathcal{L}[v] = 0$. This follows from linearity of the operator:

$$\mathcal{L}[v] = \mathcal{L}[c_1u_1 + c_2u_2] = \mathcal{L}[c_1u_1] + \mathcal{L}[c_2u_2] = c_1\mathcal{L}[u_1] + c_2\mathcal{L}[u_2] = 0\,.$$

Similarly if u_h satisfies $\mathcal{L}[u_h] = 0$ and u_p satisfies $\mathcal{L}[u_p] = f$, then $v = u_p + cu_h$ also satisfies $\mathcal{L}[v] = f$ (Exercise: prove this).

For a majority of the class, we will be discussing solutions to linear partial differential equations with constant coefficients.

1.3 Solutions to differential equations

A function u is solution to the partial differential equation

$$\mathcal{L}[u(x,y)] = f(x,y) \quad (x,y) \in D\,, \tag{18}$$

if the above equation holds for all $(x,y) \in D$.

In ODE land, we know that an nth order linear differential equation has an n parameter family of solutions. For example, the second order differential equation

$$u'' + 4u = 0\,,$$

has a two parameter family of solutions

$$u = c_1 \sin(2x) + c_2 \cos(2x)\,,$$

where c_1, c_2 are constants determined based on either the initial values or the boundary values of u.

Let us look at the nature of solutions to partial differential equations. Consider the following partial differential equation for the unknown function $u(x,y)$

$$u_{xx} - u = 0\,, \quad (x,y) \in \mathbb{R}^2\,. \tag{19}$$

It is straight forward to verify that

$$u(x,y) = f_1(y)e^x + f_2(y)e^{-x}\,, \tag{20}$$

where the functions $f_1(y)$ and $f_2(y)$ are arbitrary functions.

Let us look at another example. The wave equation in different coordinates can be rewritten as

$$u_{xy} = 0\,, \quad (x,y) \in \mathbb{R}^2\,. \tag{21}$$

Integrating the above equation in x we get

$$u_y = f(y)\,.$$

Integrating the equation in y, we get

$$u(x,y) = \int^y f(s)ds + f_2(x)\,.$$

Relabelling the $\int^y f(s)ds = f_1(y)$, the above solution can be written as

$$u(x,y) = f_1(y) + f_2(x)\,.$$

Even in this case, we see that the solution contains two arbitrary functions $f_1(y)$ and $f_2(x)$. Thus in PDEs, we have arbitrary functions which describe the general solution as opposed to arbitrary constants for ODEs. However, just as in the case of ODEs, these aribtrary functions are fixed by a combination of initial and/or boundary values of the unknown function u.

2 Lec 2

In linear ODEs, a combination of initial and boundary conditions were used to pick out particular solutions from a given family of solutions. These conditions often had physical interpretation. For example, the dynamics of a spring mass system is governed by the differential equation

$$x'' + k^2 x = 0 \tag{22}$$

The above differential equation has a two parameter solutions given by

$$x(t) = c_1 \cos(kt) + c_2 \sin(kt) \tag{23}$$

By prescribing the initial position, and inital velocity of the particle

$$x(0) = x_0 \quad \text{and} \quad v(0) = x'(0) = v_0\,,$$

we get the particular solution

$$x(t) = x_0 \cos(kt) + \frac{v_0}{k}\sin(kt)\,. \tag{24}$$

Alternatively, the governing equation for the potential in a wire of length 1 is given by

$$\frac{d}{dx}\varepsilon\frac{du}{dx} = f(x)\,, 0 < x < 1 \tag{25}$$

where $\varepsilon(x)$ is the dielectric constant of the wire and $f(x)$ is the given charge density. In this setup, a particular solution is obtained by prescribing the potential (boundary conditions) at both ends of the wire

$$u(0) = u_0\,, \quad u(1) = u_1\,.$$

Similarly, to make a problem *well-posed* in PDEs, auxilliary conditions in the form of initial and boundary conditions need to be specified. In this section, we discuss the types of boundary conditions that typically appear in second order PDEs. To discuss this, we consider three sample PDEs:

- Wave equation in the exterior of an object

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}) \quad (x, y, z) \in \mathbb{R}^2 \setminus D, \quad t_0 < t < \infty \tag{26}$$

- The heat equation in the interior of a domain D

$$u_t = \nabla \cdot \kappa \nabla u\,, \quad (x, y) \in D \quad 0 < t < \infty\,, \tag{27}$$

  where u is the temperature of the object.

- The equation for electrostatics in the composite domain $D = D_1 \cup D_2$ shown below:

$$u_{xx} + u_{yy} = 0 \quad (x, y) \in D_1\,, \tag{28}$$

$$u_{xx} + u_{yy} = 0 \quad (x, y) \in D_2\,. \tag{29}$$

  Here u represents the potential associated with the Electric field.

Let S denote the surface of the object D and $\boldsymbol{n}$ denote the outward normal to the boundary S.

2.1 Initial conditions

Definition 2.1. *An initial condition specifies the unknown function at a particular time t_0.*

Typically, if the equation has n time derivatives, then n initial conditions need to be prescribed. For example, in the wave equation above, two initial conditions corresponding to the position of the particle and its initial velocity need to be prescribed.

$$u(x, y, z, t_0) = \phi(x, y, z)\,, \quad u_t(x, y, z, t_0) = \psi(x, y, z)\,.$$

By the same reasoning, for the heat equation, we just need to prescribe the initial temperature of the object

$$u(x, y, t_0 = 0) = T_0(x, y)\,.$$

2.2 Boundary conditions

Just like in ODEs, we can also specify boundary conditions. Boundary conditions for PDEs are usually specified associated to the spatial variables. As the name suggests, the boundary conditions are specified on the boundary S of the object D. Two of the commonly speficied boundary conditions are: a) Dirichlet conditions, b) Neumann conditions.

Definition 2.2. *A Dirichlet boundary condition is when the unknown function in the PDE is specified on the boundary of the object.*

For example, in the wave equation, the position of the boundary could be specified at all times in order for the problem to be well posed

$$u(x, y, z, t) = g(x, y, z, t) \quad , (x, y, z) \in S \quad t_0 < t < \infty\,.$$

This corresponds to a Dirichlet boundary condition. Similarly, for the heat equation, the object could be surrounded by a heat bath which maintains the temperature of the boundary at a fixed value:

$$u(x, y, t) = h(x, y, t) \quad (x, y) \in S \quad 0 < t < \infty\,.$$

And in electrostatics, the potential on the boundary could be specified. If the material 1 is grounded then the potential on the boundary is uniformly 0

$$u(x, y) = 0 \quad (x, y) \in S_1\,.$$

Definition 2.3. *A Neumann boundary condition corresponds to a prescription of the normal derivative of the unknown function u*

$$u_n = \frac{\partial u}{\partial \boldsymbol{n}} = \nabla u \cdot \boldsymbol{n} \quad x \in S\,.$$

For the heat equation, the Neumann boundary condition corresponds to heat flux through the boundary. Thus, if the object is insulated on the boundary, the boundary condition for the heat equation would be

$$u_n(x, y, t) = 0 \quad (x, y) \in S, \quad 0 < t < \infty\,.$$

For the heat equation, either the Dirichlet OR the Neumann conditions need to be specified. Alternatively, the Robin boundary condition can also be specified for the heat equation.

Definition 2.4. *The robin boundary condition corresponds to a linear combination of the Dirichlet and the Neumann boundary conditions:*

$$\alpha u(x, y, t) + \beta u_n(x, y, t) = f(x, y, t) \quad (x, y) \in S \quad 0 < t < \infty\,.$$

2.3 Transmission conditions

At material interfaces, like in the electrostatics example; the value of the potential or the electric field is usually unknown. However, what is physically known in such situations is both the potential and the electric field are continuous in the whole domain $D = D_1 \cup D_2$. This results in the following boundary conditions at the interface $S_2 = D_1 \cap D_2$

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0 \\ \boldsymbol{x}\in D_1}} u(\boldsymbol{x}) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0 \\ \boldsymbol{x}\in D_2}} \quad \forall \boldsymbol{x}_0 \in S_2$$

We will use the following notation to represent the above equation

$$[[u]]_{S_2} = 0\,,$$

where $[[\phi]]_\Gamma$ denotes the jump in ϕ across the interface Γ. Similarly, the continuity of the electric field implies

$$[[\varepsilon u_n]]_{S_2} = 0$$

where $\varepsilon(\boldsymbol{x}) = \varepsilon_1$ for $\boldsymbol{x} \in D_1$ and $\varepsilon(\boldsymbol{x}) = \varepsilon_2$ for $\boldsymbol{x} \in D_2$ are the known material properties. Such boundary conditions are often referred to as transmission boundary conditions or jump conditions

2.4 Radiation conditions

Finally, for unbounded domains like the wave equation described above, the behaviour of the solution as $\boldsymbol{x} \to \infty$ also needs to be specified. For example, the Sommerfeld radiation condition given by

$$\lim_{r\to\infty} \left(\frac{\partial u}{\partial r} - \frac{\partial u}{\partial t} \right) = 0\,,$$

imposes that the waves be radiating outward to ∞.

3 Lec 3

In the previous section, we mentioned that sufficient number of boundary conditions are required in order to make a PDE well posed. Here we formally define *well-posedness.*

Definition 3.1. *Well-posedness. A PDE defined on a domain with the prescribed set of boundary conditions (these may be a combination of the types of boundary conditions described above or even completely different boundary conditions) is well-posed if it satisfies the following conditions:*

- *Existence: There exists at least one solution $u(\boldsymbol{x}, t)$ satisfying all the conditions (both the PDE in the volume and the boundary conditions)*
- *Uniqueness: There exists at most one solution*
- *Stability or continuous dependence on data: The unique solution $u(\boldsymbol{x}, t)$ depends continuously on the data of the problem, i.e., small perturbations in the data lead to small perturbations in the solution.*

Why do we care about well-posedness? At the end of the day, the partial differential equations we write down are models for describing physical phenomenon that we “observe" which quantifiably agrees with the measurements. Moreover, these measurements have errors owing to the resolution of the devices used to obtain them. Similarly, the boundary data available for the PDE has similar measurement errors. Thus, since we cannot distinguish data or measurements which are slightly perturbed, it is desirable that the corresponding solutions and hence the computed measurements from the solution of the PDEs also only be slightly perturbed.

Let us look at an example. Consider the wave equation in frequency domain:

$$u_{xx} + u_{yy} + \omega^2 u(x, y) = f(x, y) \quad (x, y) \in \mathbb{R}^2 \setminus D\,,$$

with the boundary conditions

$$u(x, y) = h(x, y) \quad (x, y) \in \partial D\,,$$

and the radiation conditions at ∞

$$\lim_{r\to\infty} r^{0.5} \left(\frac{\partial u}{\partial r} - iku \right) = 0\,.$$

The data for this problem consists of two functions $f(x, y) \in \mathbb{R}^2 \setminus D$ and $h(x, y)$ defined on the boundary ∂D. The above problem, i.e. the PDE along with the boundary condition and the radiation condition at ∞ is well-posed. We will show through the course of this class that the problem indeed has a unique solution for a given $f(x,y)$ and $h(x,y)$ (under reasonable assumptions on their smoothness, i.e. assuming f,h have $\ell > 2$ continuous derivatives). Let us illustrate, what we mean by stability. Assume $u(x,y)$ and $v(x,y)$ satisfy the following PDEs

$$u_{xx} + u_{yy} + \omega^2 u(x,y) = f_1(x,y) \quad (x,y) \in \mathbb{R}^2 \setminus D\,,$$

with the boundary conditions

$$u(x,y) = h_1(x,y) \quad (x,y) \in \partial D\,,$$

and the radiation conditions at ∞

$$\lim_{r\to\infty} r^{0.5}\left(\frac{\partial u}{\partial r} - iku\right) = 0\,.$$

$$v_{xx} + v_{yy} + \omega^2 v(x,y) = f_2(x,y) \quad (x,y) \in \mathbb{R}^2 \setminus D\,,$$

with the boundary conditions

$$v(x,y) = h_2(x,y) \quad (x,y) \in \partial D\,,$$

and the radiation conditions at ∞

$$\lim_{r\to\infty} r^{0.5}\left(\frac{\partial v}{\partial r} - ikv\right) = 0\,.$$

Then if f_1 is close to f_2 in the sense that

$$\|f_1 - f_2\|_{\mathbb{L}^\infty(\mathbb{R}^2\setminus D)} = \max_{(x,y)\in\mathbb{R}^2\setminus D} |f_1(x,y) - f_2(x,y)| < \varepsilon\,,$$

and h_1 is close to h_2

$$\|h_1 - h_2\|_{\mathbb{L}^\infty(\partial D)} = \max_{(x,y)\in\partial D} |h_1(x,y) - h_2(x,y)| < \varepsilon\,,$$

then the solutions u and v are also close to each other

$$\|u - v\|_{\mathbb{L}^\infty(\mathbb{R}^2\setminus D)} = \max_{(x,y)\in\mathbb{R}^2\setminus D} |u(x,y) - v(x,y)| \le C(\|f_1 - f_2\|_{\mathbb{L}^\infty(\mathbb{R}^2\setminus D)} + \|h_1 - h_2\|_{\mathbb{L}^\infty(\partial D)}) \le 2C\varepsilon$$

where C is a constant independent of f_1, f_2, h_1, h_2 and hence u and v.

Unfortunately, not all problems arising in physics are well-conditioned. For example, consider the same problem above with $f(x,y) = 0$. In some applications in medical imaging the goal is to detect the boundary of an obstacle. So the domain D and its boundary are unknown ∂D.

$$u_{xx} + u_{yy} + \omega^2 u = 0 \quad (x,y) \in \mathbb{R}^2 \setminus D\,,$$

$$u(x,y) = h(x,y) \quad (x,y) \in \partial D\,,$$

$$\lim_{r\to\infty} r^{0.5}\left(\frac{\partial u}{\partial r} - iku\right) = 0\,.$$

However, what is avaiable is the measurement of the solution $u_{\ell,m}(R\cos(\theta_j), R\sin(\theta_j))$ for a collection of angles θ_j, several boundary conditions $h_\ell(x,y)$, and several frequencies ω_m^2. Given these measurements, the goal is to determine the boundary of D. This problem is known to be ill-posed. Several domains D that are not "close" result in exteremly close measurements $u_{\ell,m}(R\cos(\theta_j)), R\sin(\theta_j))$. While the "forward problem" of computing the solution $u(x,y)$ given the domain D and the boundary conditions $h(x,y)$ is well-posed, the "inverse" problem of determining D given the measurements $u_{\ell,m}(R\cos(\theta_j), R\sin(\theta_j)$, boundary condition h_ℓ is ill-posed.

3.1 Waves in 1D

We now turn our attention to solving PDEs. We start off with the wave equation in 1-dimensional space over all of $\mathbb{R}$. The governing equation is

$$u_{tt} - c^2 u_{xx} = 0 \quad x \in \mathbb{R} \quad t > 0\,.$$

Defining the equation over the whole real line allows us to skip the complexity of handling boundary condition at the same time allows us to build some insight into the nature of solutions to the wave equation. In mathematics, it is natural to construct simple analytic solutions to a given partial differential equation and then use these as building blocks for more complicated problems. And so we proceed.

We will construct the solution using two methods. First, being the method of characteristics. The PDE can be decomposed into a system of two first order differential equations.

$$u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0\,.$$

Setting

$$(\partial_t + c\partial_x) = u_t + cu_x = v(x,t)\,,$$

Plugging $v(x,t)$ in the equation above, we get

$$v_t - cv_x = (\partial_t - c\partial_x)v = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0\,.$$

Thus, we get the following system of first order partial differential equations for $u(x,t)$ and $v(x,t)$.

$$v_t - cv_x = 0\,, \quad u_t + cu_x = v(x,t)\,.$$

We can solve the first equation for $v(x,t)$, using the method of characteristics discussed before.

$$v_t - cv_x = \nabla v(x,t) \cdot (-c, 1) = 0\,,$$

i.e. the directional derivative of $v(x,t)$ in the direction $(-c,1)$ is 0 – thus $v(x,t)$ is constant along the direction $(-c,1)$ in the (x,t) plane. Thus the general solution to

$$v_t - cv_x = 0\,,$$

is given by

$$v(x,t) = f(x+ct)\,,$$

Exercise 3.1. *Verify that this indeed is a solution to $v_t - cv_x = 0$ for any differentiable f*

for an arbitrary differentiable function f. Plugging it into the equation for $u(x,t)$, we get

$$u_t + cu_x = f(x+ct)\,,$$

We construct a solution to this PDE in two stages. First, we compute a particular solution $u^p(x,t)$ which satisfies

$$u^p_t + cu^p_x = f(x+ct)\,,$$

to which we add a general solution $u^h(x,t)$ to the homogeneous problem

$$u^h_t + cu^h_x = 0\,.$$

The solution $u(x,t)$ is then given by

$$u(x,t) = u^h(x,t) + u^p(x,t)\,.$$

Proceeding as above, it is straightforward to construct the homogeneous solution to the problem

$$u^h_t + cu^h_x = 0\,.$$

In this case, the function $u^h(x,t)$ is constant on lines in the direction $(c,1)$ in the (x,t) plane. Thus, a general solution to the above PDE is given by

$$u^h(x,t) = g(x-ct)\,.$$

For the particular solution u^p, we make an ansatz of the form $u^p(x,t) = h(x+ct)$. The reason for making such an ansatz is that the differential operator $\partial_t + c\partial_x$ maps functions of the form $h(x+ct)$ to themself. Plugging it into the differential equation, we get

$$(\partial_t + c\partial_x)u^p = (\partial_t + c\partial_x)h(x+ct) = \partial_t(x+ct)\cdot h'(x+ct) + \partial_x(x+ct)\cdot h'(x+ct) = 2ch'(x+ct) = f(x+ct)\,.$$

Thus, if h satisfies the ode

$$2ch'(s) = f(s)\,,$$

then $u^p(x+ct) = h(x+ct)$ is a particular solution to the PDE

$$u^p_t + cu^p_x = f(x+ct)\,.$$

Combining all of this, we see that the general solution to the wave equation on the line is

$$u(x,t) = h(x+ct) + g(x-ct)\,,$$

for arbitrary twice differentiable functions h and g.

Exercise 3.2. *Formally verify that this is indeed a solution to the PDE.*

We see that the solution has two waves, a wave going to the left at speed c corresponding to $h(x+ct)$, and a wave going to the right at speed c corresponding to $g(x-ct)$. It is easy to see that by setting $h(s)$ to be a bump function centered at the origin. At time $t=0$, $h(x+ct) = h(x)$ corresponds to a bump centered at the origin, at time $t=1$, the solution $h(x+ct) = h(x+c)$ is a bump centered at $-c$.

Let's rederive the same solution using a different technique. From the discussion above, we notice that the directions $x+ct$ and $x-ct$ play an important role in the solution to the differential equation. So let us change coordinates to $\xi = x+ct$ and $\eta = x-ct$. Using the chain rule

$$\partial_x = \partial_\xi \cdot \frac{\partial \xi}{\partial x} + \partial_\eta \cdot \frac{\partial \eta}{\partial x} = \partial_\xi + \partial_\eta \,,$$

and

$$\partial_t = \partial_\xi \cdot \frac{\partial \xi}{\partial t} + \partial_\eta \cdot \frac{d\eta}{dt} = c\partial_\xi - c\partial_\eta$$

Then combining these partial derivatives, we get

$$\partial_t + c\partial_x = 2c\partial_\xi \,, \quad \partial_t - c\partial_x = -2c\partial_\eta \,.$$

The wave equation in the new coordinates can be written as

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = (2c\partial_\xi)\cdot(-2c\partial_\eta)u = -c^2 u_{\xi\eta} = 0 \,.$$

The solution to the above equation is given by

$$u = h(\xi) + g(\eta) = h(x+ct) + g(x-ct) \,,$$

which is exactly what we had before.

The arbitrary functions can be determined if we are given initial conditions for the problem. To be consistent with the text, I'll relabel the general solution of the wave equation on the real line:

$$u_{tt} - c^2 u_{xx} = 0 \quad x \in \mathbb{R} \,,, \quad t > 0 \,,$$

to

$$u(x,t) = f(x+ct) + g(x-ct) \,. \tag{30}$$

Now, we wish to determine functions f and g such that $u(x,t)$ satisfies the initial conditions:

$$u(x,0) = \phi(x) \,, \quad u_t(x,0) = \psi(x) \,.$$

We wish to find functions f, g in terms of ϕ, ψ. Plugging in $t=0$ in equation (30) we get

$$u(x,0) = f(x) + g(x) = \phi(x) \,.$$

Differentiating equation (30) with respect to t we get

$$\partial_t u(x,t) = cf'(x+ct) - cg'(x-ct) \,.$$

Plugging in $t=0$, we get

$$\partial_t u(x,0) = cf'(x) - cg'(x) = \psi(x) \,.$$

Thus, we need to solve this system of equations to obtain the unknown functions f, g in terms of the given functions ϕ, ψ.

Exercise 3.3. *Show, that the solution to the above system of equations is infact given by*

$$f(s) = \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi + A \,.$$

and

$$g(s) = \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi + B \,.$$

Since $f+g=\phi$, we conclude that $A+B=0$ and the solution to the initial value problem is given by

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds \tag{31}$$

Exercise 3.4. *Verify that this indeed is a solution to the initial value problem:*

$$u_{tt} - c^2 u_{xx} = 0 \quad x \in \mathbb{R} \,, \quad t > 0 \,, \quad u(x,0) = \phi(x) \,, \quad u_t(x,0) = \psi(x) \,.$$

4 Lec 4

4.1 Properties of solutions to wave equation in 1D

4.1.1 Finite speed of propogation

The solution to the wave equation can be expressed as waves travelling in either direction with speed c

$$u(x,t) = f(x+ct) + g(x-ct) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds\,.$$

This property has a further consequence, Information travels at finite speed in this case at a sspeed $\leq c$. This is also referred to as *the principle of causality.*

Furthermore, from the analytic form of the solution, it is clear that the initial data at $(x_0, 0)$, i.e. $\phi(x_0)$ and $\psi(x_0)$; at time t only influences the solution between $x_0 - ct < x < x_0 + ct$. This region as a function of t is referred to as the *domain of influence* of the point x_0. Another way to state the same result is that if $\phi(x), \psi(x) = 0$ for $|x| > R$, then at time t the solution is still 0 for $|x| > R + ct$, i.e. the domain of influcence of the segment $|x| \leq R$ is the sector $|x| \leq R + ct$.

A related concept is that of the *domain of dependence*. What segment of the intial data does $u(x,t)$ depend on? This segment is referred to as the *domain of dependence*. Again, it is clear from the expression of the solution above that the solution depends on intial data supported in $[x-ct, x+ct]$, more particularly $\psi(s)$ for $s \in [x-ct, x+ct]$, $\phi(x+ct)$ and $\phi(x-ct)$.

4.1.2 Conservation of energy

Consider an infinite string with density ρ and tension T. The wave equation in this set up is given by

$$\rho u_{tt} = T u_{xx} \quad x \in \mathbb{R} \quad t > 0\,.$$

In physics, we know that the kinetic energy of an object is given by:

$$KE(t) = \frac{1}{2}mv^2 = \frac{1}{2}\rho\int_{-\infty}^{\infty} u_t^2(x,t)dx\,,$$

since the velocity of the particle at location x is the rate of change of displacement u with respect to time which is u_t. The potential energy of a stretched spring is given by

$$PE(t) = \frac{1}{2}k(\Delta x)^2 = \frac{1}{2}T\int_{-\infty}^{infty} u_x^2(x,t)dx\,,$$

where u_x denotes the stretching the string.

In the wave equation, the total energy of the spring is given by

$$E(t) = KE(t) + PE(t) = \frac{1}{2}\rho\int_{-\infty}^{\infty} u_t^2(x,t)dx + \frac{1}{2}T\int_{-\infty}^{\infty} u_x^2(x,t)dx\,,$$

is conserved. It is an empty statement if the energy is not finite to begin with. So we will make a simplifying assumption to ensure that the initial energy is finite. We will assume that ϕ and ψ are supported on $|x| \leq R$. Thus,

$$E(0) = \frac{1}{2}\rho\int_{-\infty}^{\infty} \psi(x)^2 dx + \frac{1}{2}T\int_{-\infty}^{\infty} \phi_x^2 dx < \infty\,.$$

To proved that the energy is conserved, we will show that the rate of change of energy is zero.

$$\begin{aligned}
\frac{dE}{dt} &= \frac{1}{2}\frac{d}{dt}\rho\int_{-\infty}^{\infty} u_t^2(x,t) + \frac{1}{2}T\frac{d}{dt}\int_{-\infty}^{\infty} u_x^2\,, && (32)\\
&= \frac{1}{2}\rho\int_{-\infty}^{\infty} \frac{d}{dt}u_t^2 + \frac{1}{2}T\int_{-\infty}^{\infty}\frac{d}{dt}u_x^2\,, && (33)\\
&= \rho\int_{-\infty}^{\infty} u_t u_{tt} dx + T\int_{-\infty}^{\infty} u_x u_{tx} && (34)\\
&= \int_{-\infty}^{\infty} u_t \rho u_{tt} dx + T u_x u_t|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} T u_{xx} u_t\,, \quad \text{Integration by parts} && (35)\\
&= \int_{-\infty}^{\infty} u_t(\rho u_{tt} - T u_{xx})dx = 0\,, \quad \text{Since } u(x,t) \text{ is supported on } |x| \leq R + ct. && (36)
\end{aligned}$$

The above result also implies that $E(t) = E(0)$ for all t.

4.1.3 Uniqueness

The conservation of energy gives us a straightforward proof of uniqueness of solutions to the wave equation on the line. Consider two solutions of the wave equation with the same initial data $u^1(x,t)$ and $u^2(x,t)$. Both these functions satisfy

$$u^1_{tt} = c^2 u^1_{xx}\,, \quad u^1(x,0) = \phi(x)\,, \quad u^1_t(x,0) = \psi(x)\,.$$

$$u^2_{tt} = c^2 u^2_{xx}\,, \quad u^2(x,0) = \phi(x)\,, \quad u^2_t(x,0) = \psi(x)\,.$$

Then, their difference $w = u^1 - u^2$ also satisfies the wave equation but with zero initial data

$$w_{tt} = u^1_{tt} - u^2_{tt} = c^2(u^1_{xx} - u^2_{xx}) = c^2 w_{xx}\,,$$

$$w(x,0) = u^1(x,0) - u^2(x,0) = \phi(x) - \phi(x) = 0\,.$$

$$w_t(x,0) = u^1_t(x,0) - u^2_t(x,0) = \psi(x) - \psi(x) = 0\,.$$

Thus, w is a solution to the wave equation. However, $E(0) = 0$ since $w_t = w_x = 0$ for all $x \in \mathbb{R}$ for $t = 0$ (initial conditions for w). From energy conservation principle, we conclude that $E(t) = 0$ for all t, i.e.

$$\rho \int w_t(x,t)^2 + T \int w_x(x,t)^2 = 0\,.$$

This implies that $w_t = w_x = 0$ for all x and all t. Thus, $w(x,t)$ is a constant and is 0 since it is 0 at $t = 0$.

Thus, the wave equation cannot have two distinct solutions with the same initial data.

At this stage, we've constructed solutions to show existence, we've proven uniqueness. Stability of solutions to the wave equation on the real line will be on HW1.

4.2 Diffusion in 1D

The diffusion equation in one dimension is given by

$$u_t - k u_{xx} = f(x,t) \quad x \in \mathbb{R}\,, \quad u(x,0) = \phi(x)\,.$$

where $f(x,t)$ and $\phi(x)$ are known functions, and $k > 0$ is the diffusion constant. The process of deriving a solution to the above problem is fairly involved so we will postponed the discussion to later. However, in the same spirit as that of the wave equation, we study elementary properties of the solutions of the diffusion equation.

To simplify things even further, we are going to restrict our attention the diffusion equation on a line segment $0 < x < \ell$. The PDE is then given by

$$u_t - k u_{xx} = f(x,t)\,, \quad 0 < x < \ell, \quad t > 0$$

$$u(x,0) = \phi(x)\,, \quad u(0,t) = f(t)\,, \quad u(\ell,t) = g(t)\,.$$

4.2.1 Maximum and minimum principle

Solutions to the diffusion/heat equation with no source term, i.e. $f(x,t) = 0$ for all x and t satisfy the maximum principle. Let u be a solution to the diffusion equation with no source term, i.e.

$$u_t = k u_{xx} \quad 0 < x < \ell \quad 0 < t \leq T\,.$$

$$u(x,0) = \phi(x)\,, \quad u(0,t) = f(t)\,, \quad u(\ell,t) = g(t)$$

Then $u(x,t)$ achieves its maxium on one of the boundaries, $t = 0$, $x = 0$ or $x = \ell$, i.e

$$\max_{\substack{0\leq x\leq \ell \\ 0\leq t\leq T}} u(x,t) = \max_{\{t=0\}\cup\{x=0\}\cup\{x=\ell\}} u(x,t) = \max\{\max \phi(x), \max f(t), \max g(t)\} \tag{37}$$

The intuitive explanation is the following. Suppose (x_0,t_0) is the location of the maximum of $u(x,t)$. Then $u_t(x_0,t_0) = 0$, and $u_x(x_0,t_0) = 0$. Furthermore, $u_{xx}(x_0,t_0) \leq 0$. If we could show that $u_{xx} < 0$ and not ≤ 0, then we are done since we would have a contradiction in $u_t = k u_{xx}$ as the left hand side is 0 and the right hand side is not. To convert the above intuition into a proof, we buy ourselves a little wiggle room. Consider $v(x,t) = u(x,t) + \varepsilon x^2$ for an arbitrary $\varepsilon > 0$. Then

$$v_t - k v_{xx} = u_t - u_{xx} - k\varepsilon \frac{\partial}{\partial x}\frac{\partial}{\partial x} x^2 = 0 - 2k\varepsilon = -2k\varepsilon < 0$$

As discussed above, if (x_0,t_0) is a local maximum of $v(x,t)$, then $v_t(x_0,t_0) = 0$ and $v_{xx}(x_0,t_0) \leq 0$. Thus, $v_t(x_0,t_0) - v_{xx}(x_0,t_0) \geq 0$, which contradicts $v_t - v_{xx} = -2k\varepsilon < 0$.

We still haven't ruled out the top lid (x,T).

Exercise 4.1. *Show that $v(x,t)$ cannot achieve its maximum at (x_0,T) using a similar argument.*

Thus, $v(x,t)$ achieves its maximum on one of the boundaries $t=0, x=0$ or $x=\ell$. If M is the maximum of u on these boundaries, then $\max v(x,t) = M+\varepsilon\ell^2$, and $v(x,t) \leq M+\varepsilon\ell^2$, for all $0\leq x\leq \ell$, $0\leq t\leq T$.) From the definition of v,

$$u(x,t)+\varepsilon x^2 = v(x,t) \leq M+\varepsilon\ell^2 \implies u(x,t)\leq M+\varepsilon(\ell^2-x^2) \quad \forall x\in[0,\ell] \quad t\in[0,T]$$

Since, ε in the above equation is arbitrary, we conclude that $u(x,t)\leq M$ in the entire domain and thus the result.

It is very important to note that u satisfies the diffusion equation with no source term, i.e. $u_t = ku_{xx}$, or $f(x,t)=0$.

From the maximum principle, it is straightforward to derive the minimum principle for $u(x,t)$ by considering $-u(x,t)$ instead of $u(x,t)$

Exercise 4.2. *Assuming the maximum principle to be true, prove the minimum principle.*

4.2.2 Uniqueness

Given the maximum and the minimum principle, it is straightforward to show uniqueness for the heat equation on the line segment. Consider two distinct solutions u^1 and u^2 to the diffusion equation with the same initial condition $\phi(x)$ and the same boundary data $f(t), g(t)$. Then, by linearity, the difference $w=u^1-u^2$ satisfies

$$w_t - kw_{xx} = u^1_t - u^2_t - k(u^1_{xx}-u^2_{xx}) = 0\,.$$

$$w(x,0) = u^1(x,0)-u^2(x,0) = \phi(x)-\phi(x) = 0\,.$$
$$w(0,t) = u^1(0,t)-u^2(0,t) = f(t)-f(t) = 0\,.$$
$$w(\ell,t) = u^1(\ell,t)-u^2(\ell,t) = g(t)-g(t) = 0\,.$$

The maximum and minimum of w on the boundaries $t=0$,$x=0$ and $x=\ell$ are 0. By the maximum principle, the maximum is achieved on the boundary. Thus $w(x,t)\leq 0$ for all x,t in the domain. Similarly, the minimum is also achieved on the boundary. Thus $w(x,t)\geq 0$ for all x,t in the domain. Therefore, $w(x,t)=0$ for all x,t which proves the result.

We prove uniqueness, using an alternate energy principle argument. However, instead of physically arguing an energy conservation, we take a more mathematical approach this time around. We multiply the equation $w_t - kw_{xx}$ by w and integrate for all x at a fixed time t.

$$0 = 0\cdot w = ww_t - wkw_{xx}$$

. Integrating between $x=0$ to $x=\ell$, we get

$$0 = \int_0^\ell ww_t - kww_{xx}dx = \int_0^\ell \frac{1}{2}(w^2)_t - kww_{xx}dx\,, \tag{38}$$

$$0 = \int_0^\ell \frac{1}{2}(w^2)_t - kww_x|_0^\ell + \int_0^\ell kw_x^2\,, \quad \text{Integration by parts} \tag{39}$$

$$\frac{1}{2}\frac{d}{dt}\int_0^\ell w^2dx = -\int_0^\ell kw_x^2dx \leq 0\,, \quad \text{Since } w=0 \text{ for } x=0 \text{ and } x=\ell \tag{40}$$

Thus, $\int_0^\ell w(x,t)^2dx$ is a positive decreasing function of t. Since, $\int_0^\ell w(x,0)^2dx = 0$, we conclude that $\int_0^\ell w(x,t)^2 = 0$ for all t and hence $w(x,t)=0$ for all $0\leq x\leq\ell$ and all t in the domain.

5 Lec 5

In this section, we will derive a solution for the diffusion or heat equation on the real line

$$u_t = ku_{xx} \qquad -\infty < x < \infty\,, \quad t>0 \tag{41}$$

$$u(x,0) = \phi(x) \tag{42}$$

As before, we proceed with constructing simple solutions to the differential equation and construct a solution to the initial value problem by using these simple solutions as builiding blocks.

5.1 Properties of solutions to the diffusion equation

Before we proceed to construct solutions to the differential equation, let us study some interesting properties of solutions to the PDE.

1. *Spatial and temporal translation invariance of the PDE*:
   Spatial invariance: If $u(x,t)$ is a solution to the PDE, then for a fixed y, $u(x-y,t)$ is also a solution to the PDE. Physically, if $u(x,t)$ represents a "heat" source placed at the origin $(0,0)$, then $u(x-y,t)$ represents a heat source placed at $(y,0)$. *Temporal invariance*: If $u(x,t)$ is a solution to the PDE, then for a fixed τ, $u(x,t-\tau)$ also is a solution to the PDE. In the same spirit as above, $u(x,t-\tau)$ would represent a heat source placed at $(0,\tau)$.

   Exercise 5.1. *Verify that the heat equation satisfies the spatial and temporal invariance property.*

2. *Linear combinations*: Since the heat equation is a linear homogeneous PDE, linear combinations of solutions are still solutions to the differential equation. Combining the spatial invariance and linear combination property – if $u(x,t)$ is a solution to the PDE, then
$$\sum_{j=1}^{N} c_j u(x-y_j,t)\,,$$
   where c_j are constants also represents a solution to the heat equation.

3. *Intergral solutions*: A limiting argument of the linear combination case would show that if $S(x,t)$ represents a solution to the differential equation, then
$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy\,,$$
   also represents a solution to the differential equation for a "nice" enough function $g(x)$ defined on the real line. Here by nice, we mean the set of functions for which the improper integral on the right hand side converges. Neglecting the issue of convergence for now, if we were allowed to switch the derivative and the integral we get:
$$v_t = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} S(x-y,t)g(y)dy = \int_{-\infty}^{\infty} S_t(x-y,t)g(y)dy\,.$$
   Similarly,
$$v_{xx} = \frac{\partial}{\partial x}\frac{\partial}{\partial x}\int_{-\infty}^{\infty} S(x-y,t)g(y)dy = \int_{-\infty}^{\infty} S_{xx}(x-y,t)g(y)dy\,.$$
   Then,
$$v_t - kv_{xx} = \int_{-\infty}^{\infty} (S_t(x-y,t) - kS_{xx}(x-y,t))g(y)dy = 0\,, \quad \text{Since } S(x-y,t) \text{ satifies heat equation.}$$

4. *Dilation between spatial and temporal variables.* Another common technique for constructing more solutions to the differential equation is to rescale the spatial and temporal variables. Suppose $u(x,t)$ represents a solution to the heat equation, we seek α such that $v(x,t) = u(\lambda^\alpha x, \lambda t)$ is also a solution to the heat equation for any λ.
$$v_t(x,t) = \frac{\partial}{\partial t}u(\lambda^\alpha x,\lambda t) = \lambda u_t(\lambda^\alpha x,\lambda t)\,,$$
$$v_{xx}(x,t) = \frac{\partial}{\partial x}\frac{\partial}{\partial x}u(\lambda^\alpha x,\lambda t) = \lambda^{2\alpha}u_{xx}(\lambda^\alpha x,\lambda t)\,.$$
   For v to satisfy the heat equation, we require
$$0 = v_t - kv_{xx} = \lambda u_t - k\lambda^{2\alpha}u_{xx}\,.$$
   Thus, if $\lambda^{2\alpha} = \lambda$, i.e., $\alpha = 0.5$, $v(x,t) = u(\sqrt{\lambda}x, \lambda t)$ satisfies the heat equation if $u(x,t)$ satisfies the heat equation, for all $\lambda > 0$. An interesting consequence of the property is the following lemma.

   Lemma 5.1. *if $\phi(x)$ is dilation invariant, i.e. $\phi(\sqrt{\lambda}x) = \phi(x)$ for all λ, then the corresponding solution $u(x,t)$ satisfies $u(\sqrt{\lambda}x, \lambda t) = u(x,t)$ for all $\lambda > 0$.*

   Proof. Let $u(x,t)$ be a solution to the initial value problem:
$$u_t = ku_{xx}\,, \quad u(x,0) = \phi(x)\,. \tag{43}$$

Let $\lambda > 0$ and consider the initial value problem:

$$v_t = kv_{xx} \quad v(x,0) = \phi(\sqrt{\lambda}x)\,. \tag{44}$$

From the dilation property, $v(x,t) = u(\sqrt{\lambda}x, \lambda t)$ also satisfies the heat equation and has boundary data $v(x,0) = u(\sqrt{\lambda}x, 0) = \phi(\sqrt{\lambda}x)$. Since, $\phi(x) = \phi(\sqrt{\lambda}x)$, the two initial value problems (43) and (44) are identical. By uniqueness of solutions to the heat equation, we conlude that $u(x,t) = v(x,t)$ for all x, t. $\therefore u(x,t) = u(\sqrt{\lambda}x, \lambda t)$ for all $\lambda > 0$. $\square$

A further interesting consequence of the above lemma is the following lemma which we state without proof.

Lemma 5.2. *If $u(x,t)$ satisfies the heat equation and $u(\sqrt{\lambda}x, \lambda t) = u(x,t)$ then $u(x,t) = g\left(c\frac{x}{\sqrt{t}}\right)$ for any constant c*

It is straightforward to show that if

$$u(x,t) = g\left(c\frac{x}{\sqrt{t}}\right),$$

then $u(x,t)$ satisfies the dilation property. This follows by

$$u(\sqrt{\lambda}x, \lambda t) = g\left(c\frac{\sqrt{\lambda}x}{\sqrt{\lambda t}}\right) = g\left(c\frac{x}{\sqrt{t}}\right) = u(x,t)\,.$$

Showing the result in the other direction is a little tricky for which we will essentially construct a solution to the heat equation and appeal to uniqueness of solutions to initial value problems of the heat equation.

5.2 Green's function

Based on the properties discussed above, we will look for a solution to the diffusion/heat equation:

$$u_t = ku_{xx}\,, \quad u(x,0) = \phi(x)\,, \tag{45}$$

of the form

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy\,,$$

where $S(x,t) = \partial_x Q(x,t)$ and $Q(x,t)$ satisfies the heat equation

$$Q_t = kQ_{xx}\,, \quad Q(x,0) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}.$$

To ignore issues of convergence for the time being, we assume that $\phi(x)$ is compactly supported, i.e. $\phi(x) = 0$ for all $|x| > R$. Why do we choose to represent the solution in this form? Well, let us just illustrate that if we find such a solution Q, then

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)\,,$$

satisfies the heat equation (45) By property 3 discussed above, we see that $u(x,t)$ does satisfy the heat equation. Now all we need to worry about is the initial data of $u(x,t)$. For this we shall use one of the favorite theorems of a PDE theorist, integration by parts

$$\begin{aligned} u(x,t) = \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)dy &= -\int_{-\infty}^{\infty} \frac{\partial Q}{\partial y}(x-y,t)\phi(y)dy \\ &= -Q(x-y,t)\phi(y)|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy \\ &= \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy \quad \text{Since } \phi \text{ is compactly supported} \end{aligned}$$

Plugging in $t = 0$ in the above equation, we get

$$\begin{aligned} u(x,0) &= \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)dy \\ &= \int_{-\infty}^{x} Q(x-y,0)\phi'(y) + \int_{x}^{\infty} Q(x-y,0)\phi'(y)dy \\ &= \int_{-\infty}^{x} 1\cdot\phi'(y) + \int_{x}^{\infty} 0\cdot\phi'(y)dy \\ &= \phi(x) \end{aligned}$$

Thus, given such a particular solution to the heat equation, we can solve any initial value problem for the heat equation. The initial data for $Q(x,t)$ i.e. $Q(x,0) = 1$ for $x > 0$ and 0 for $x < 0$, satisfies the dilation property. Thus, using property 5 discussed above, we may look for a solution of the form

$$Q(x,t) = g\left(\frac{x}{\sqrt{4kt}}\right).$$

Thus, we have reduced the task of finding a function of two variables to a function of one variable and that will end up converting the PDE to an ODE. Following the derivation in the text in section 2.4, we see that if $Q(x,t)$ satisfies the heat equation, then g satisfies the ordinary differential equation:

$$g''(p) + 2pg' = 0.$$

Using your favorite method to solve the above ODE, we conclude that

$$g'(p) = c_1 e^{-p^2},$$

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp,$$

$$S(x,t) = \frac{1}{2\sqrt{\pi kt}} e^{-\frac{x^2}{4kt}} \quad t > 0,$$

and finally

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)dy.$$

$S(x,t)$ defined above, is what is refered to as the Green's function or the fundamental solution to the heat equation in one dimension, or alternatively as the source function or the propogator for the diffusion equation. At time t, $S(x,t)$ is a Gaussian with variance $4kt$. Thus, for small t the bulk of $S(x,t)$ is concentrated around the origin and for large t, $S(x,t)$ is essentially flat. Furthermore

$$\int_{-\infty}^{\infty} S(x,t)dx = 1 \quad \forall t > 0.$$

Thus, the Green's function for the heat equation is essentially an averaging operator. The solution at time t and location x is the weighted average of the initial data $\phi(x)$. For small t, the weights are concentrated in a small vicinity of x where as for large t, the weights are evenly spread out. The physical interpretation is as follows: suppose you heat up a section of a conducting rod to a higher temperature. Then say $\phi(x) = 1$ for $-\delta < x < \delta$ and 0 otherwise. Then as time progresses, the heat will essentially spread out to the whole rod till the temperature is uniform. Thus, we see that even non smooth initial data smoothens out due to the averaging phenomenon. Infact as t increases, the smoothness of the function $u(x,t)$ as a function of x increases.

6 Lec 6

6.1 Reflections for the heat equation

We now consider a simple prototype of solutions to heat equation with boundary wherein the solutions can be constructed using elementary techniques. The methods discussed here are in fact analytical tools used to develop more efficient numerical solvers for a large class of problems.

Consider the heat equation on half of the real line:

$$u_t = ku_{xx} \quad 0 < x < \infty, \quad \text{PDE,} \tag{46}$$
$$u(x,0) = \phi(x), \quad 0 < x < \infty, \quad \text{Initial conditions,} \tag{47}$$
$$u(0,t) = 0, \quad 0 < t < \infty, \quad \text{Boundary conditions.} \tag{48}$$

What goes wrong if we try to proceed as before. The solution of the heat equation on the whole real line is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left(\frac{-(x-y)^2}{4kt}\right) \phi(y)dy.$$

Since $\phi(x)$ in now defined on $(0,\infty)$ as opposed to $(-\infty,\infty)$ earlier, one might be tempted to look for a solution to the initial-boundary value problem of the form

$$u_1(x,t) = \int_0^{\infty} S(x-y,t)\phi(y)dy = \frac{1}{\sqrt{4\pi kt}} \int_0^{\infty} \exp\left(\frac{-(x-y)^2}{4kt}\right) \phi(y)dy.$$

For the above representation, the PDE is clearly satisfied. Proceeding as in Section 5, the initial condition $u(x,0) = \phi(x)$ is also satisfied. However, the boundary condition,

$$u_1(0,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \exp\left(\frac{-y^2}{4kt}\right)\phi(y)dy\,,$$

is not satisfied. Thus, we need to add in a solution, which still satisfies the PDE, does not alter the initial conditions but annihiliates the boundary condition generated by $u_1(x,t)$.

One technique commonly used to achieve this task, particular for simple boundaries such as half planes or circles, is called the method of reflections. The idea is simple, the physical domain of the problem is $0 < x < \infty$, $0 < t < \infty$. We will place "auxilliary" heat sources with strategically chosen strengths so as to annihiliate the solution $u_1(x,t)$ on the boundary $(0,t)$.

The kernel $S(x,t)$ is symmetric in x about the origin, i.e., it is an even function of x. The effect of the heat source placed at $(x,0)$ to the solution at $(0,t)$ is identical to the effect of the heat source placed at $(-x,0)$ to the solution $(0,t)$ – $S(x,t) = S(-x,t)$. Thus, a simple way to annihiliate the solution $u_1(x,t)$, would be to add in the solution due to an initial condition $-\phi(x)$ at $(-x,0)$, i.e.,

$$u(x,t) = \int_0^\infty S(x-y,t)\phi(y)dy + \int_0^\infty S(x+y,t)\cdot(-\phi(y))dy = \frac{1}{\sqrt{4\pi kt}}\int_0^\infty \left(\exp\left(\frac{-(x-y)^2}{4kt}\right) - \exp\left(\frac{-(x+y)^2}{4kt}\right)\right)\phi(y)dy\,. \tag{49}$$

Another way to interpret the result above is the following: Consider the odd extension of the function $\phi(x)$ given by

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & x > 0\,, \\ -\phi(-x) & x < 0\,, \\ 0 & x = 0\,. \end{cases}$$

Consider the initial value problem

$$u_t = ku_{xx}\,, \quad -\infty < x < \infty \quad 0 < t < \infty\,, \tag{50}$$

$$u(x,0) = \phi_{\text{odd}}(x)\,, \quad -\infty < x < \infty\,. \tag{51}$$

The solution $v(x,t)$ to the above initial value problem is exactly the solution given in equation (49). Let us verify this.

$$\begin{aligned} v(x,t) &= \int_{-\infty}^\infty S(x-y,t)\phi_{\text{odd}}(y)dy\,, \\ &= -\int_{-\infty}^0 S(x-y,t)\phi(-y)dy + \int_0^\infty S(x-y,t)\phi(y)dy\,, \\ &= -\int_0^\infty S(x+y,t)\phi(y)dy + \int_0^\infty S(x-y,t)\phi(y)dy\,. \end{aligned}$$

We now claim that $u(x,t)$ solves the initial-boundary value problem given by equations (46),(47), and (48).

As discussed above, $u(x,t)$ also satisfies the initial value problem given by equations (50), and (51). Thus, $u(x,t)$ satisfies the PDE, and $u(x,0) = \phi_{\text{odd}}(x)$. In particular, $u(x,0) = \phi(x)$ for $x > 0$. Finally,

$$u(0,t) = \int_0^\infty S(-y,t)\phi(y) - \int_0^\infty S(y,t)\phi(y) = 0 \quad \text{(Since } S \text{ is even)}\,.$$

7 Lec 7

7.1 Reflections for the wave equation

We now turn our attention to a similar initial-boundary value problem for the wave equation defined on the half line given by:

$$v_{tt} = c^2 v_{xx}\,, \quad 0 < x < \infty\,, \quad 0 < t < \infty \quad \text{PDE}\,, \tag{52}$$

$$v(x,0) = \phi(x)\,, \quad v_t(x,0) = \psi(x)\,, \quad 0 < x < \infty\,, \quad \text{Initial conditions}\,, \tag{53}$$

$$v(0,t) = 0\,, \quad 0 < t < \infty\,, \quad \text{Boundary conditions.} \tag{54}$$

For the heat equation defined on the half line, we saw that using an odd-extension for the initial condition annihiliates the solution $v(x,t)$ on the line $x = 0$. Proceeding in a similar fashion, we consider the following initial value problem:

$$\begin{aligned} v_{tt} &= c^2 v_{xx}\,, \quad -\infty < x < \infty\,, \quad 0 < t < \infty\,, \\ v(x,0) &= \phi_{\text{odd}}(x)\,, \quad v_t(x,0) = \psi_{\text{odd}}(x)\,, \quad 0 < t < \infty\,. \end{aligned}$$

where ϕ_{odd} and ψ_{odd} are the odd extensions of the functions ϕ and ψ respectively. The solution to this initial value problem is given by

$$v(x,t) = \frac{1}{2}[\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(s)ds\,. \tag{55}$$

Equation (55) is infact also a solution to our initial boundary value problem given by equations (52), (53), and (54). $v(x,t)$ satisfies the PDE and initial conditions given by (52), and (53), since it also satisfies the auxilliary initial value problem with initial conditions ϕ_{odd} and ψ_{odd} respectively. Furthermore,

$$v(0,t) = \frac{1}{2}[\phi_{\text{odd}}(ct) + \phi_{\text{odd}}(-ct)] + \frac{1}{2c}\int_{-ct}^{ct} \psi_{\text{odd}}(s)ds = 0\,,$$

since $\phi_{\text{odd}}(ct) = -\phi_{\text{odd}}(-ct)$ and $\int_{-a}^{a} f(s)ds = 0$ for any odd function f.

7.2 Duhamel's principle

We now turn our attention to solving the inhomogeneous initial value problems for both the heat and the wave equations. Before, we proceed to obtaining a solution to the inhomogeneous PDEs, we turn our attention to Duhamel's principle.

Lemma 7.1. *Duhamel's principle. Consider the following constant coefficient nth order ordinary differential equation:*

$$\mathcal{L}[y](t) = y^{(n)} + a_{n-1}y^{(n-1)}(t) + \ldots + a_0 y(t) = f(t)\,. \tag{56}$$

with initial data

$$y(0) = 0\,, y'(0) = 0 \ldots y^{(n-1)}(0) = 0\,.$$

Suppose $w(t)$ is a solution to the **homogeneous** *problem with initial data*

$$w(0) = 0\,, w'(0) = 0\,, \ldots w^{(n-2)}(0) = 0\,, w^{(n-1)}(0) = 1\,.$$

(The choice of the initial data for w will be clarified in the proof.) Then the solution to the inhomogeneous problem with given boundary conditions is

$$y(t) = \int_0^t w(t-s)f(s)ds\,.$$

Proof. Taking a derivative of y in the above expression, we get

$$y'(t) = \frac{d}{dt}\int_0^t w(t-s)f(s)ds = w(t-t)f(t) + \int_0^t \frac{dw}{dt}(t-s)f(s)ds = \int_0^t \frac{dw}{dt}(t-s)f(s)ds\,, \quad \text{Since } w(0)=0.$$

Using an inductive argument, it is easy to show that

$$y^{(j)}(t) = \int_0^t w^{(j)}(t-s)f(s)ds\,, \quad \forall j = 0,1,2\ldots n-1\,. \tag{57}$$

Exercise 7.1. *Fill in the details for the inductive argument above.*

From Equation (57), it is clear that $y(t)$ satisfies the initial conditions:

$$y(0) = 0\,, y'(0) = 0 \ldots y^{(n-1)}(0) = 0\,.$$

Taking the derivative $w^{(n-1)}$, we get

$$y^{(n)}(t) = \frac{d}{dt}\int_0^t w^{(n-1)}(t-s)f(s)ds = w^{(n-1)}(t-t)f(t) + \int_0^t w^{(n)}(t-s)f(s)ds = f(t) + \int_0^t w^{(n)}(t-s)f(s)ds\,, \quad \text{Since } w^{(n-1)}(0)=1.$$

Finally,

$$\begin{aligned}
\mathcal{L}[y](t) &= y^{(n)} + a_{n-1}y^{(n-1)}(t) + \ldots a_0 y(t) \\
&= f(t) + \int_0^t w^{(n)}(t-s)f(s)ds + \int_0^t a_{n-1}w^{(n-1)}(t-s)f(s)ds + \ldots \int_0^t a_0 w(t-s)f(s)ds \\
&= f(t) + \int_0^t \mathcal{L}[w](t-s)f(s)ds \\
&= f(t) \quad (\mathcal{L}[w] = 0 \text{ as } w \text{ satisfies the homogeneous ODE.})
\end{aligned}$$

$\square$

What does the above lemma tell us? The response for a forcing function $f(s)$ supported on a infinitesimally small time interval $[0, ds]$ is roughly given by

$$y(t) \approx w(t)f(ds)ds \quad 0 < t < ds$$

i.e., a solution of the homogeneous problem with initial data for an *impulse*

$$w(0) = 0\,, w'(0) = 0\,, \ldots w^{(n-2)}(0) = 0\,, w^{(n-1)}(0) = 1\,.$$

Furthermore, given a solution to the homogeneous problem with impulse initial conditions, the solution of the inhomogeneous problem is given by a convolution of the forcing function and the solution $w(t)$. Another way to interpret the inhomogeneous solution obtained using Duhamel's principle is the following: The solution to the inhomogeneous ODE (56) given by

$$y(t) = \int_0^t w(t-s)f(s)\,ds = \int_0^t \mathcal{P}^s[f](t)ds\,,$$

where $\mathcal{P}^s[f](t)$ is the operator which given a function f defined on $0 < t < \infty$ on output returns the solution to the initial value problem

$$\mathcal{L}[y] = 0\,, \quad y^{(j)}(s) = 0 \quad j = 0, 1, \ldots n-2, \quad y^{(n-1)}(s) = f(s)\,,$$

i.e., we propogate the ODE with initial data $f(s)$ for time $t-s$. Since, in our case, the ode is a constant coefficient linear ODE, by uniqueness

$$\mathcal{P}^s[f](t) = w(t-s)f(s)\,.$$

This abstract principle naturally extends to solutions of inhomogeneous constant coefficient PDEs as well.

7.3 Inhomogeneous heat equation

Consider the inhomogeneous heat equation given by:

$$u_t - ku_{xx} = f(x,t)\,, \quad -\infty < x < \infty\,, \quad 0 < t < \infty\,.$$

$$u(x,0) = \phi(x)\,.$$

Using linearity of the PDE, we split the task of computing the solution into two parts:

$$u(x,t) = u_h(x,t) + u_p(x,t)\,,$$

where the solution to the homogeneous problem $u_h(x,t)$ is responsible for the initial data and satisfies:

$$\partial_t u_h - k\partial_{xx} u_h = 0\,, \quad u_h(x,0) = \phi(x)\,,$$

and the particular solution $u_p(t)$ satisfies

$$\partial_t u_p - k\partial_{xx} u_p = f(x,t)\,, \quad u_p(x,0) = 0\,.$$

We have already derived the solution of the homogeneous problem to be

$$u_h(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy\,.$$

Using Duhamel's principle then, the particular solution to the inhomogeneous problem is given by

$$u_p(x,t) = \int_0^t \mathcal{P}^s[f](x,t)ds\,,$$

where $\mathcal{P}^s[f](x,t)$ is the solution of the initial value problem:

$$u_t = ku_{xx} \quad -\infty < x < \infty\,, \quad 0 < t\,,$$

with initial conditions

$$u(x,s) = f(x,s)\,,$$

i.e., we propogate the solution to the homogeneous heat equation with initial data $f(y,s)$ for time $t-s$ to obtain the solution $\mathcal{P}^s[f](x,t)$. The solution to the above initial value problem is given by:

$$\mathcal{P}^s[f](x,t) = \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dy$$

Thus,

$$u_p(x,t) = \int_0^t \mathcal{P}^s[f](x,t)ds = \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds = \int_0^t \frac{1}{\sqrt{4\pi k(t-s)}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{(4k(t-s))}\right) f(y,s)dyds\,.$$

Let us formally verify that u_p is indeed a solution to the inhomogeneous problem.

$$\begin{aligned}
\partial_t u_p(x,t) &= \partial_t \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds\,, \\
&= \int_0^t \int_{-\infty}^{\infty} \partial_t S(x-y,t-s)f(y,s)dyds + \lim_{s\to t} \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dy \\
&= \int_0^t \int_{-\infty}^{\infty} k\partial_{xx} S(x-y,t-s)f(y,s)dyds + \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} S(x-y,\varepsilon)f(y,t-\varepsilon)dy \quad \text{(Since } S(x,t) \text{ satisfies heat equation)} \\
&= k\partial_{xx} u_p(t) + \lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} S(x-y,\varepsilon)f(y,t-\varepsilon)dy \\
&= k\partial_{xx} u_p(t) + f(x,t)
\end{aligned}$$

The last equation follows from the observation that the initial data corresponding to the solution of the heat equation

$$\lim_{t\to 0} \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy\,,$$

is $\phi(x)$.

7.4 Inhomogeneous wave equation

We now turn our attention to the inhomogeneous wave equation given by:

$$u_{tt} - c^2 u_{xx} = f(x,t)\,, -\infty < x < \infty\,, 0 < t\,.$$

$$u(x,0) = \phi(x)\,, \quad u_t(x,0) = \psi(x)\,.$$

Using linearity of the PDE, we split the task of computing the solution into two parts:

$$u(x,t) = u_h(x,t) + u_p(x,t)\,,$$

where the solution to the homogeneous problem $u_h(x,t)$ is responsible for the initial data and satisfies:

$$\partial_{tt} u_h - c^2 \partial_{xx} u_h = 0\,, \quad u_h(x,0) = \phi(x)\,, \quad \partial_t u_h(x,0) = \psi(x)\,,$$

and the particular solution $u_p(t)$ satisfies

$$\partial_{tt} u_p - c^2 \partial_{xx} u_p = f(x,t)\,, \quad u_p(x,0) = 0\,, \quad \partial_t u_p(x,0) = 0.$$

We have already derived the solution of the homogeneous problem to be

$$u_h(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds\,.$$

Using Duhamel's principle then, the particular solution to the inhomogeneous problem is given by

$$u_p(x,t) = \int_0^t \mathcal{P}^s[f](x,t)ds\,,$$

where $\mathcal{P}^s[f](x,t)$ is the solution of the initial value problem:

$$u_{tt} = c^2 u_{xx} \quad -\infty < x < \infty\,, \quad 0 < t\,,$$

with initial conditions

$$u(x,s) = 0\,, \quad u_t(x,s) = f(x,s)\,,$$

i.e., we propogate the solution to the homogeneous wave equation with initial velocity $f(y,s)$ for time $t-s$ to obtain the solution $\mathcal{P}^s[f](x,t)$. The solution to the above initial value problem is given by:

$$\mathcal{P}^s[f](x,t) = \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy$$

Thus,

$$u_p(x,t) = \int_0^t \mathcal{P}^s[f](x,t)ds = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy\,ds\,.$$

Let us now verify that $u_p(x,t)$ indeed satisfies the inhomogeneous equation with the 0 initial conditions. It is straight-forward to check that $u_p(x,0) = 0$ for all x. Differentiating under the integral sign can be tricky in this setup. Let

$$G(x,t,s) = \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy\,.$$

Then

$$u_p(x,t) = \frac{1}{2c}\int_0^t G(x,t,s)ds\,.$$

And now we are in business.

$$\partial_t u_p(x,t) = \frac{1}{2c}\left[G(x,t,t) + \int_0^t \partial_t G(x,t,s)\right]$$

$$\partial_t G(x,t,s) = \partial_t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy \;= \int_0^t \left(c\cdot f(x+c(t-s),s) - (-c)\cdot f(x-c(t-s),s)\right)ds$$

$$\therefore\, \partial_t u_p(x,t) = \frac{1}{2c}\left[0 + \int_0^t \left(c\cdot f(x+c(t-s),s) - (-c)\cdot f(x-c(t-s),s)\right)ds\right] = \frac{1}{2}\int_0^t \left(f(x+c(t-s),s) + f(x-c(t-s),s)\right)ds$$

From the expression above, we readily see that

$$\partial_t u_p(x,0) = 0\,.$$

Taking one more derivative, we get

$$\begin{aligned}
\partial_{tt} u_p(x,t) &= \partial_t\left[\frac{1}{2}\int_0^t \left(f(x+c(t-s),s) + f(x-c(t-s),s)\right)ds\right]\\
&= \frac{1}{2}\left[f(x+c(t-t),t) + f(x-c(t-t),t) + \int_0^t \partial_t\left(f(x+c(t-s),s) + f(x-c(t-s),s)\right)ds\right]
\end{aligned}$$

$$\partial_t f(x+c(t-s),s) = \partial_x f(x+c(t-s),s)\cdot\partial_t(x+c(t-s)) = \partial_x f(x+c(t-s),s)\cdot c$$
$$\partial_t f(x-c(t-s),s) = \partial_x f(x-c(t-s),s)\cdot\partial_t(x-c(t-s)) = \partial_x f(x-c(t-s),s)\cdot(-c)$$
$$\partial_{tt} u_p(x,t) = f(x,t) + \frac{c}{2}\cdot\int_0^t \left(f(x+c(t-s),s) - f(x-c(t-s),s)\right)ds$$

Exercise 7.2. *Proceed as above to show that*

$$\partial_{xx} u_p(x,t) = \frac{1}{2c}\int_0^t \left(f(x+c(t-s),s) - f(x-c(t-s),s)\right)ds\,.$$

Combining these results, we conclude that u_{pp} satisfies the inhomogeneous wave equation with forcing function $f(x,t)$ and the 0 initial conditions.

7.4.1 Well-posedness

We have already shown existence and uniqueness for the inhomogeneous wave equation. T he proof for uniqueness is same as that for the homogeneous problem, and existence follows from the formula we derived above. The last ingredient to show that the inhomogeneous problem is well-posed is stability. For this purpose, we consider the wave equation on the finite time interval $t \in [0,T]$. We will show stability in the $\mathbb{L}^\infty$ or the supremem norm, i.e

$$\|f\|_{\mathbb{L}^\infty(\mathbb{R}\times[0,T])} = \sup_{\substack{x\in(-\infty,\infty)\\ t\in[0,T]}} |f(x,t)|\,, \quad \|\phi\|_{\mathbb{L}^\infty(\mathbb{R})} = \sup_{x\in(-\infty,\infty)} |\phi(x)|\,.$$

The statement of stability of solutions to the inhomogeneous wave equation would be

$$\|u(x,t)\|_{\mathbb{L}^\infty(\mathbb{R}\times[0,T])} \leq C_1(T)\|f(x,t)\|_{\mathbb{L}^\infty(\mathbb{R}\times[0,T])} + C_2(T)\|\phi(x)\|_{\mathbb{L}^\infty(\mathbb{R})} + C_3(T)\|\psi(x)\|_{\mathbb{L}^\infty(\mathbb{R})}\,,$$

where the constants C_1, C_2, and C_3 are independent of the functions f, ϕ, ψ, and u and only possibly depend on T. The solution to the inhomogeneous initial value wave equation is given by

$$u_h(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds\,.$$

A straight forward estimate shows that the above result holds with constants

$$C_1(T) = \frac{T^2}{2}\,, \quad C_2(T) = 1\,, \quad C_3(T) = T\,.$$

The reason the above result implies stability is the following. Let u_1 and u_2 be solutions to the inhomogeneous intial value wave equation with forcing functions f_1, and f_2, and initial values $\phi_1(x), \psi_1(x)$ and $\phi_2(x), \psi_2(x)$ respectively. Suppose the functions f_1 and f_2 are "close" in the distance defined above, i.e.

$$\|f_1 - f_2\|_{\mathbb{L}^\infty(\mathbb{R}\times[0,T])}\,.$$

Similarly, suppose that the initial values are close in the sense

$$\|\phi_1 - \phi_2\|_{\mathbb{L}^\infty(\mathbb{R})}\,, \quad \|\psi_1 - \psi_2\|_{\mathbb{L}^\infty(\mathbb{R})}\,.$$

Then, the above result tells us that the maximum difference between u_1 and u_2 is also small.

7.5 Proofs of convergence for the diffusion equation

The proof discussed above is just a formal proof of $u(x,t)$ satisfying the inhomogeneous heat equation. In fact, the proof that

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} S(x-y,\varepsilon)\phi(y)dy = \phi(x)\,,$$

i.e., the solution to the homogeneous initial value problem for the heat equation given by

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\,,$$

achieves the right initial value was also formal and not rigorous. Furthermore, even the fact that $u(x,t)$ above solves the heat equation was also formal. In this section, we discuss the tools necessary to make both of these proofs rigorous, i.e. that $u(x,t)$ defined above does indeed solve the heat equation and satisfies the right initial conditions. Similar arguments can be used to make the proofs for the inhomogeneous case rigorous and will be discussed in practice problem set 3.

We first note that the solution $u(x,t)$ is expressed as a convolution of $S(\cdot,t)$ and $\phi(\cdot)$,

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy = \int_{-\infty}^{\infty} S(z,t)\phi(x-z)\,,dz\,.$$

Lemma 7.2. *If $\phi(x)$ is a bounded continuous function, then*

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy\,,$$

satisifes the heat equation for $x\in\mathbb{R}$ and $t>0$. Furthermore, $u(x,t)$ is a smooth function, i.e. it has continuous partial derivatives of all orders for all $t>0$.

Proof. We know that $S(x-y,t)$ satisfies the heat equation for all $x\in\mathbb{R}$ and $t>0$. Taking the time derivative of $u(x,t)$, we get

$$\partial_x u(x,t) = \partial_x \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy\,.$$

We can switch the order of differentiation and integration as long as the following integrals converge absolutely

$$\int_{-\infty}^{\infty} S(x-y,t)dy\phi(y) \quad \text{and} \quad \int_{-\infty}^{\infty} \partial_x S(x-y,t)\phi(y)dy\,.$$

The first part is easy to show

$$\int_{-\infty}^{\infty} |S(x-y,t)\phi(y)|dy \le \max_{y\in\mathbb{R}}|\phi(y)|\int_{-\infty}^{\infty} S(x-y,t)dy = \max_{y\in\mathbb{R}}|\phi(y)|$$

The first inequality follows from triangle inequality for integrals and using the positivity of the kernel $S(x-y,t)$ and the second one follows from the fact that $\int_{\mathbb{R}} S(x-y,t)dy = 1$ for all $x\in\mathbb{R}$ and $t>0$. The second part is a slightly more tedious calculation, but proceeds in a similar fashion

$$\int_{-\infty}^{\infty} |\partial_x S(x-y,t)\phi(y)|dy = \int_{-\infty}^{\infty}\left|\frac{1}{\sqrt{4\pi kt}}\frac{(x-y)}{2kt}\exp\left(\frac{-(x-y)^2}{4kt}\right)\phi(y)\right|dy \tag{58}$$

$$= \frac{c}{\sqrt{t}}\int_{-\infty}^{\infty}|p|\exp\left(\frac{-p^2}{4}\right)|\phi(x-p\sqrt{kt}| \quad \text{Change of variable: } p = \frac{(x-y)}{\sqrt{kt}} \tag{59}$$

$$\le \max_{y\in\mathbb{R}}|\phi(y)|\frac{c}{\sqrt{t}}\int_{-\infty}^{\infty}|p|\exp\left(-\frac{p^2}{4}\right) \tag{60}$$

which converges.

Exercise 7.3. *Show that the integrals*

$$\int_{-\infty}^{\infty} |p|^n exp - \frac{p^2}{4},$$

converge for all n.

Thus

$$\partial_x \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy = \int_{-\infty}^{\infty} \partial_x S(x-y,t)\phi(y)dy.$$

and the integral on the right converges uniformly. Similarly, taking higher orer derivatives in x or p results in integrands of the form

$$\max_{y\in\mathbb{R}} |\phi(y)| c(t) \int_{-\infty}^{\infty} |p|^n exp - \frac{p^2}{4},$$

where $c(t) < \infty$. Thus $u(x,t)$ has partial derivatives of all orders and the derivatives commute with the integral. Furthermore, since $S(x-y,t)$ solves the heat equation for $t > 0$, we conclude that $u(x,t)$ also solves the heat equation for $t > 0$ since,

$$\partial_t u(x,t) - k\partial_{xx} u(x,t) = \partial_t \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy - k\partial_{xx} \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy = \int_{-\infty}^{\infty} (\partial_t - k\partial_{xx}) S(x-y,t)\phi(y)dy = 0$$

□

Lemma 7.3. *If* $\phi(\cdot)$ *is continuous at* x *and* $\max_{y\in\mathbb{R}} |\phi(y)| < \infty$*, then*

$$\lim_{t\to 0} \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy = \phi(x).$$

Proof. Since $\int_{-\infty}^{\infty} S(x-y,t) = 1$, consider

$$\phi(x) - \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy \int_{-\infty}^{\infty} S(x-y,t)(\phi(x) - \phi(y))dy,$$

Using the change of variables in the proof above, the above integral can be rewritten as

$$\frac{1}{\sqrt{4\pi}} \int_{\infty}^{\infty} \exp\left(-\frac{p^2}{4}\right)(\phi(x) - \phi(x - p\sqrt{kt}))dp$$

We wish to show that

$$\lim_{t\to 0} \frac{1}{\sqrt{4\pi}} \int_{\infty}^{\infty} \exp\left(-\frac{p^2}{4}\right)(\phi(x) - \phi(x - p\sqrt{kt}))dp = 0$$

The intuition to show the above result is the following, as $t \to 0$, $\phi(x - p\sqrt{4kt}$ gets closer and closer to $\phi(x)$ except for really large values of p, owing to the continuity of ϕ at x. However, for large values of p, $\exp -\frac{p^2}{4}$ ends up driving the integral to 0. Following this intuition, we break $p \in \mathbb{R}$ into two regimes, $|p\sqrt{kt}| \le \delta$ where δ is such that $|\phi(x) - \phi(y)| \le \varepsilon/2$ for $|x-y| \le \delta$ and the second regime being $|p| \ge \frac{\delta}{\sqrt{kt}}$

$$|\frac{1}{\sqrt{4\pi}} \int_{\infty}^{\infty} \exp\left(-\frac{p^2}{4}\right)(\phi(x) - \phi(x - p\sqrt{kt}))dp| \le \frac{1}{\sqrt{4\pi}} \int_{|p\sqrt{kt}|\le\delta} \exp\left(-\frac{p^2}{4}\right)|(\phi(x) - \phi(x - p\sqrt{kt}))|dp + \tag{61}$$

$$\frac{1}{\sqrt{4\pi}} \int_{|p|\ge\delta/\sqrt{kt}} \exp\left(-\frac{p^2}{4}\right)|(\phi(x) - \phi(x - p\sqrt{kt}))|dp \tag{62}$$

$$\le \int_{|p\sqrt{kt}\le\delta} \exp\left(-\frac{p^2}{4}\right) \frac{\varepsilon}{+} \int_{|p|\ge\delta/\sqrt{kt}} \exp\left(-\frac{p^2}{4}\right)|(\phi(x) - \phi(x - p\sqrt{kt}))|dp \tag{63}$$

$$\le \varepsilon/2 + 2\max_{y\in\mathbb{R}} |\phi(y)| \int_{|p|\ge\delta/\sqrt{kt}} \exp\left(-\frac{p^2}{4}\right)dp \tag{64}$$

$$\le \varepsilon/2 + \varepsilon/2 \tag{65}$$

Given an ε, we have already picked $\delta > 0$. Now given this δ, we can choose t small enough so that

$$2\max_{y\in\mathbb{R}} |\phi(y)| \int_{|p|\ge\delta/\sqrt{kt}} \exp\left(-\frac{p^2}{4}\right)dp = C\int_{|p|>N} \exp\left(-\frac{p^2}{4}\right)dp \le \varepsilon/2.$$

This follows from the fact that the integral

$$\int_{-\infty}^{\infty} \exp\left(-\frac{p^2}{4}\right)dp,$$

converges.

□

8 Separation of variables

We now turn our attention to solving PDEs on finite intervals.

8.1 Wave equation

Consider the initial value problem for wave equation with homogeneous dirichlet conditions on the finite interval

$$u_{tt} = c^2 u_{xx}\,, \quad 0 < x < \ell \tag{66}$$

$$u(0,t) = 0 = u(\ell,t) \tag{67}$$

$$u(x,0) = \phi(x)\,, \qquad u_t(x,0) = \psi(x)\,. \tag{68}$$

To simplify problem, we look for solutions of special form

$$u(x,t) = X(x)T(t)\,. \tag{69}$$

The procedure of computing such special solutions where the solution is expressed as a product of functions of each independent variable is called *separation of variables*. Clearly, not all functions of the form would satisfy a wave equation. Plugging in the representation for $u(x,t)$ in equation (74) into the PDE, we get

$$X(x)T''(t) = c^2 X''(x)T(t)\,. \tag{70}$$

Dividing by $c^2 X(x)T(t)$, we get

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)}\,.$$

Since the equation on the left is solely a function of t and the equation on the right is solely a function of x, the only way for that to be so is if both were constants, i.e.

$$\frac{T''(t)}{c^2 T(t)} = \lambda \quad \frac{X''(x)}{X(x)} = \lambda\,.$$

The experts in ODEs in this room know that there are three cases: $\lambda = \beta^2$, $\lambda = -\beta^2$, $\lambda = 0$. We leave it as an exercise below to show that $\lambda = \beta^2$ and $\lambda = 0$ result in trivial solutions i.e. $u(x,t) = 0$ for all (x,t). Let us focus on $\lambda = -\beta^2$. The solution for $X(x)$ is given by

$$X(x) = C\cos(\beta x) + D\sin(\beta x)\,.$$

The corresponding solution for $T(t)$ is given by

$$T(t) = A_n\cos(\beta ct) + B_n\sin(\beta ct)\,.$$

On imposing the boundary conditions $u(0,t) = 0 = u(\ell,t)$, we get

$$X(0)T(t) = 0 = X(\ell)T(t)\,,$$

Since $T(t) \not\equiv 0$, we conclude that $X(x)$ must satisfy the boundary conditions $X(0) = 0$ and $X(\ell) = 0$. On imposing the boundary conditions, we get

$$X(0) = C\cdot 1 + D\cdot 0 = 0 \implies C = 0\,.$$

$$X(\ell) = D\sin(\beta\ell) = 0\,.$$

To obtain a non-trivial solution, we must have $D \neq 0$, thus β must satisfy $\sin(\beta\ell) = 0$, i.e

$$\beta = \frac{n\pi}{\ell} \quad n \in \mathbb{N}\,.$$

Thus,

$$X_n(x) = \sin\left(\frac{n\pi x}{\ell}\right),$$

which are exactly the eigenfunctions for the operator $\mathcal{L}$ with homogeneous dirichlet boundary conditions given by

$$\mathcal{L}[X] = X''(x)\,, \quad X(0) = 0 = X(\ell)\,.$$

The correpsonding solution to the wave equation is given by

$$u_n(x,t) = \left(A_n\cos\left(\frac{n\pi ct}{\ell}\right) + B_n\sin\left(\frac{n\pi ct}{\ell}\right)\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

By linearity, finite linear combinations of such solutions are also solutions to the initial value problem for the wave equation with homogeneous dirichlet boundary conditions, i.e.

$$u(x,t) = \sum_{n=1}^{N} \left(A_n \cos\left(\frac{n\pi ct}{\ell}\right) + B_n \sin\left(\frac{n\pi ct}{\ell}\right)\right) \sin\left(\frac{n\pi x}{\ell}\right),$$

represents a solution to the wave equation as long as

$$\phi(x) = \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi x}{\ell}\right),$$

$$\psi(X) = \sum_{n=1}^{N} \frac{n\pi c}{\ell} B_n \sin\left(\frac{n\pi x}{\ell}\right).$$

The space of prescribed initial values that can be reached by such a family of solutions seems fairly restrictive. However, to overcome this issue, we may express the solutions as an infinite series of the form

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{\ell}\right) + B_n \sin\left(\frac{n\pi ct}{\ell}\right)\right) \sin\left(\frac{n\pi x}{\ell}\right),$$

with initial data given by

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right),$$

$$\psi(X) = \sum_{n=1}^{\infty} \frac{n\pi c}{\ell} B_n \sin\left(\frac{n\pi x}{\ell}\right),$$

as long as all the series and two derivatives of the series converge uniformly. Such a series is called a *Fourier sine series.* Turns out a fairly large family of functions can be expressed as convergent Fourier sine series. A detailed discussion of the Fourier series is postponed to a later date.

Definition 8.1. *Frequency. The co-efficients of t inside the sines and cosines in the above representation is called the frequency of the solution*

Returning back to the example of a vibrating string, like that of a guitar or violin, where both ends are clamped to enforce 0 displacement at all times, we see that the solution is expressed as a sum of harmonics or overtones, i.e sums of solutions whose frequencies are integer multiples of the fundamental frequency

$$f_0 = \frac{\pi c}{\ell}.$$

Thus, given the length of a string, we can tune a string to have a particular fundamental frequency by changing the tension or the density of the string. The wave speed c in a vibrating string with tension T and density ρ is given by $c = \sqrt{T}/\sqrt{\rho}$.

Exercise 8.1. *$\lambda \in \mathbb{C}$ is an eigenvalue of the differential operator $\mathcal{L}[X] = X''(x)$ with homogeneous dirichlet boundary conditions $X(0) = 0 = X(\ell)$, if there exists a non-zero solution to the boundary value problem $X'' = \lambda X(x)$ with the homogeneous dirichlet boundary conditions. Just like linear operators in finite dimensions, i.e. matrices, have a finite number of eigenvalues, linear differential operators on function spaces, i.e. in infinite dimensions also have an infinite collection of eigenvalues. In fact, it is known that for linear differential operators, there exists an infinite number of eigenvalues whose only accumulation point is ∞. Back to the problem at hand. The general solution to the differential equation using (possibly) complex exponentials is given by*

$$X(x) = Ce^{\sqrt{\lambda}x} + De^{-\sqrt{\lambda}x}.$$

On imposing the boundary conditions, we obtain a linear system for C and D of the form

$$A(\lambda)\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ e^{\sqrt{\lambda}\ell} & e^{-\sqrt{\lambda}\ell} \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

λ is an eigenvalue if and only if there exists a non-zero solution to the boundary value problem at hand, i.e. both C and D cannot be zero. Thus, we need a non-zero null vector $[C, D]$ to the above system of equation. This implies that λ is an eigenvalue for the boundary value problem as long as $\det A(\lambda) = 0$ or in other words, the matrix $A(\lambda)$ is singular or the rows and columns of $A(\lambda)$ are linearly dependent. Use this, to find all possible eigenvalues of the operator $\mathcal{L}$ above.

8.2 Heat equation/Diffusion equation

Consider the initial value problem for heat equation with homogeneous dirichlet conditions on the finite interval

$$u_t = k^2 u_{xx}\,, \quad 0 < x < \ell \tag{71}$$

$$u(0,t) = 0 = u(\ell,t) \tag{72}$$

$$u(x,0) = \phi(x)\,. \tag{73}$$

As before, we look for solutions of special form

$$u(x,t) = X(x)T(t)\,. \tag{74}$$

Plugging in the representation for $u(x,t)$ in equation (74) into the PDE, we get

$$X(x)T'(t) = kX''(x)T(t)\,. \tag{75}$$

Dividing by $kX(x)T(t)$, we get

$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}\,.$$

Proceeding as before, we set

$$\frac{X''(x)}{X(x)} = -\beta^2\,, \quad \frac{T'(t)}{T(t)} = -k\beta^2\,.$$

The solution for $X(x)$ is given by

$$X(x) = C\cos(\beta x) + D\sin(\beta x)\,.$$

The corresponding solution for $T(t)$ is given by

$$T(t) = A_n \exp(-\beta^2 kt)\,.$$

On imposing the boundary conditions $u(0,t) = 0 = u(\ell,t)$, we see that the non-trivial solutions to the heat equation on the finite interval are obtained when

$$\beta = \frac{n\pi}{\ell} \quad n \in \mathbb{N}\,,$$

and the corresponding solution is given by The correpsonding solution to the heat equation is given by

$$u_n(x,t) = A_n \exp\left(-\frac{n^2\pi^2 kt}{\ell^2}\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

By linearity, finite or infinite (without worrying about convergence) linear combinations of such solutions are also solutions to the initial value problem for the heat equation with homogeneous dirichlet boundary conditions, i.e.

$$u(x,t) = \sum_{n=1}^{\infty} A_n \exp\left(-\frac{n^2\pi^2 kt}{\ell^2}\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

represents a solution to the heat equation as long as

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right),$$

8.3 Neumann boundary conditions

We can find similar expansions for the Neumann boundary conditions too. Let us first consider the wave equation with Neumann boundary conditions,

$$u_{tt} = c^2 u_{xx}\,, \quad 0 < x < \ell \tag{76}$$

$$u_x(0,t) = 0 = u_x(\ell,t) \tag{77}$$

$$u(x,0) = \phi(x)\,, \qquad u_t(x,0) = \psi(x)\,. \tag{78}$$

In this setup, $X_n(x)$ are the eigenfunctions for the operator

$$\mathcal{L}[X] = X''(x)\,, \quad X'(0) = 0 = X'(\ell)\,.$$

It can be shown that the eigenvalues of the above operator are also non-negative are given by

$$\lambda_n = -\left(\frac{n\pi}{\ell}\right)^2\,, \quad n = 1, 2\ldots$$

the corresponding eigenfunctions $X_n(x)$ are

$$X_n(x) = \cos(\lambda_n x) = \cos\left(\frac{n\pi x}{\ell}\right).$$

and the corresponding separation of variables solution to the wave equation is given by The correpsonding solution to the wave equation is given by

$$u_n(x,t) = \left(A_n \cos\left(\frac{n\pi ct}{\ell}\right) + B_n \sin\left(\frac{n\pi ct}{\ell}\right)\right) \cos\left(\frac{n\pi x}{\ell}\right)$$

For the Neumann boundary value problem above, $\lambda = 0$ is also an eigenvalue and it is easy to check that $X(x) \equiv 1$ is the corresponding eigenfunction. The solution to the differential equation for $T(t)$ is given by

$$T(t) = A + Bt\,,$$

and the corresponding solution to the wave equation is given by

$$u_0(x,t) = A + Bt\,.$$

By linearity, finite or infinite (without worrying about convergence) linear combinations of such solutions are also solutions to the initial value problem for the wave equation with homogeneous Neumann boundary conditions, i.e.

$$u(x,t) = \frac{1}{2}A_0 + \frac{1}{2}B_0 t \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{\ell}\right) + B_n \sin\left(\frac{n\pi ct}{\ell}\right)\right) \cos\left(\frac{n\pi x}{\ell}\right)$$

represents a solution to the wave equation as long as the initial conditions are expressed as a *Fourier cosine series*

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{N} A_n \cos\left(\frac{n\pi x}{\ell}\right),$$

$$\psi(X) = \frac{1}{2}B_0 + \sum_{n=1}^{N} \frac{n\pi c}{\ell} B_n \cos\left(\frac{n\pi x}{\ell}\right).$$

Thus, we see that the structure of the solution and the procedure to obtain it is exactly the same as that for the homogeneous dirichlet boundary conditions case. Abstractly, we compute eigenfunctions of the operator depending on the spatial variables and express the solution as a series expansion in the eigenfunction bases where the coefficients of the expansion depend on time.

A similar calculation can be carried for the heat equation with homogeneous Neumann boundary conditions to obtain a general separation of variables solution of the heat equation in two dimensions as

$$u(x,t) = \frac{1}{2}A_0 \sum_{n=1}^{\infty} A_n \exp\left(-\frac{n^2\pi^2 kt}{\ell^2}\right) \cos\left(\frac{n\pi x}{\ell}\right)$$

as long as

$$\phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{\ell}\right),$$

Similar calculations can be carried out for mixed boundary conditions or Schrödinger's equation and will be covered in the practice exercises.

9 Fourier series

So far, we have constructed solutions to various boundary value problems in terms of "Fourier series". In this section, we study various properties of Fourier series in a generalized sense and discuss various modes of convergence.

A Fourier series of a function $\phi(x)$ defined on the interval $[-1,1]$ (can be generalized to the interval $[a,b]$, but we will stick to $[-1,1]$ for ease of notation) is given by

$$\phi(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}\,, \tag{79}$$

To determine the coefficients c_n, we first observe that

$$\int_{-1}^{1} e^{in\pi x} e^{-im\pi x}\, dx = \int_{-1}^{1} e^{i(n-m)\pi x}\, dx = \frac{1}{i\pi(n-m)}\left[e^{i(n-m)\pi} - e^{i(m-n)\pi}\right] = 0\,.$$

Given this identity, the coefficients c_n are readily obtained via multiplying equation (79) with $e^{-in\pi x}$ and integrating from -1 to 1.

$$\begin{aligned}
\int_{-1}^{1} \phi(x) e^{-in\pi x}\, dx &= \int_{-1}^{1} e^{-in\pi x} \sum_{m=-\infty}^{\infty} c_m e^{im\pi x}\, dx \\
&= \sum_{m=-\infty}^{\infty} \int_{-1}^{1} c_m e^{i(m-n)\pi x}\, dx \\
&= c_n \int_{-1}^{1} 1\, dx + \sum_{\substack{m=-\infty \\ m\neq n}}^{\infty} c_m \int_{-1}^{1} e^{i(m-n)\pi x}\, dx \\
&= 2c_n
\end{aligned}$$

Thus, the coefficients c_n are readily given by

$$c_n = \frac{1}{2} \int_{-1}^{1} \phi(x) e^{-in\pi x}\, dx\,.$$

The above result holds in a fairly general sense.

9.1 Generalized fourier series, orthogonality, and completeness

Consider a linear differential operator $\mathcal{L}[X]$, coupled with boundary conditions $\mathcal{B}[X]$. For the homogeneous dirichlet boundary value example:

$$\mathcal{L}_D[X] = -X''(x)\,, \mathcal{B}_D[X] : X(-1) = 0 = X(1)\,.$$

Similarly, for the homogeneous neumann boundary value example:

$$\mathcal{L}_N[X] = -X''(x)\,, \mathcal{B}_N[X] : X'(-1) = 0 = X'(1)\,.$$

Note, the boundary conditons $\mathcal{B}[X]$ are a part of the definition of the operator $\mathcal{L}[X]$.

In the finite dimensional case, all linear operators are matrices. Moreover, we have a defined inner product on our vector space $\mathbb{C}^N$. Suppose a, b are vectors in $\mathbb{C}^N$, then

$$(a, b) = \sum_{n=1}^{N} a_n \bar{b}_n\,.$$

The inner product induces a norm given by

$$\|a\|_{\ell^2} = \sqrt{(a,a)}\;,.$$

Similarly on function spaces, we can define an inner product. given complex functions f and g defined on $[-1,1]$, their inner product is given by

$$(f, g) = \int_{-1}^{1} f(x)\overline{g}(x)\, dx\,.$$

Exercise 9.1. *Show that $\|f\|_{\mathbb{L}^2[-1,1]} = \sqrt{(f,f)}$ defines a norm, i.e. it satisfies the following three properties:*

1.
$$\|f\|_{\mathbb{L}^2[-1,1]} \geq 0 \quad \textit{and} \quad \|f\|_{\mathbb{L}^2[-1,1]} = 0 \iff f \equiv 0\,,$$

2.
$$\|\lambda f\|_{\mathbb{L}^2[-1,1]} = |\lambda| \|f\|_{\mathbb{L}^2[-1,1]} \quad \forall \lambda \in \mathbb{C}\,,$$

3.
$$\|f + g\|_{\mathbb{L}^2[-1,1]} \leq \|f\|_{\mathbb{L}^2[-1,1]} + \|g\|_{\mathbb{L}^2[-1,1]}\,.$$

In finite dimensional land, we know that symmetric operators A have real eigenvalues and eigenvectors corresponding to distinct eigenvalues are orthogonal to each other. Moreover symmetric/hermitian linear operators are those which satisfy

$$(x, Ay) = (y, Ax)\,.$$

Let $v_1, v_2 \ldots v_N$ denote the eigenfunctions corresponding to eigenvalues $\lambda_1, \lambda_2 \ldots \lambda_n$, then any vector v can be expanded in the eigenvector basis as

$$v = \sum_{n=1}^{N} c_n v_n\,,$$

where the coefficients c_n are given by

$$c_n = \frac{(v, v_n)}{(v_n, v_n)}\,.$$

A similar result holds in infinite dimension land. A differential operator $\mathcal{L}$ along with boundary conditions $\mathcal{B}$ is symmetric or hermitian if given any two functions $f(x)$ and $g(x)$ on $[-1, 1]$, the operator $\mathcal{L}$ satisfies

$$(f, \mathcal{L}[g]) = (\mathcal{L}[f], g)\,.$$

For now, we restrict our attention to the case where $\mathcal{L}[X] = -X''(x)$. Then

$$\begin{aligned}(f, \mathcal{L}[g]) - (\mathcal{L}[f], g) &= \int_{-1}^{1} -f(x)\overline{g}''(x) + f''(x)\overline{g}(x)\,dx\,,\\ &= -f(x)\overline{g}'(x) + f'(x)\overline{g}(x)|_{-1}^{1} \quad \text{(Integration by parts)}\end{aligned}$$

The last equation is also referred to as *Green's second identity.* Thus the operator $\mathcal{L}[X] = -X''(x)$ is symmetric/hermitian if the boundary conditions $\mathcal{B}[f]$ and $\mathcal{B}[g]$ are such that

$$-f(1)\overline{g}'(1) + f'(1)\overline{g}(1) + f(0)\overline{g}'(0) - f'(0)\overline{g}'(0) = 0\,,$$

It is easy to verify that if f and g satisfy any of homogeneous Dirichlet, homogeneous Neumann, periodic boundary conditions or robin boundary conditions then the above identity holds. Thus, the differential operator $\mathcal{L}[X] = -X''$ with homogeneous dirichlet boundary conditions (or homogeneous Neumann, or periodic, or robin boundary conditions) is hermitian.

We have an analogous theorem regarding the orthogonality of eigenfunctions, all eigenvalues being real and expansions of functions in the eigenvector basis.

Definition 9.1. *Eigenfunction. $X(x)$ is an eigenvector of the operator pair $(\mathcal{L}, \mathcal{B})$ with eigenvalue λ if it satisfies*

$$\mathcal{L}[X] = \lambda X\,,$$

and X satisfies the boundary conditions $\mathcal{B}$.

Theorem 9.1. *Given a symmetric operator and boundary condition pair $\mathcal{L}[X], \mathcal{L}[B]$ then, let $X_1(x), X_2(x)\ldots$ denote the eigenfunctions of the opertor corresponding to eigenvalues $\lambda_1, \lambda_2\ldots$:*

1. *All the eigenvalues λ_n are real*
2. *Eigenvectors X_n corresponding to distinct eigenvalues are orthogonal, i.e. $(X_n, X_m) = 0$ if $\lambda_n \neq \lambda_m$*
3. *Given an expansion of a function in the eigenfunction basis*

$$f(x) = \sum_n A_n X_n(x)\,,$$

the coefficients are given by

$$A_n = \frac{(f, X_n)}{(X_n, X_n)}\,.$$

Proof.

1. Let λ be an eigenvalue and X be the corresponding eigenfunction. Then

$$\lambda(X, X) = (\lambda X, X) = (\mathcal{L}[X], X) = (X, \mathcal{L}[X]) = (X, \lambda X) = \overline{\lambda}(X, X)\,.$$

Thus, $\lambda = \overline{\lambda}$

2. Let X_1, X_2 be eigenfunctions corresponding to distinct eigenvalues λ_1 and λ_2 respectively. Then

$$\lambda_1(X_1, X_2) = (\lambda_1 X_1, X_2) = (\mathcal{L}[X_1], X_2) = (X_1, \mathcal{L}[X_2]) = (X_1, \lambda_2 X_2) = \overline{\lambda_2}(X_1, X_2) = \lambda_2(X_1, X_2)\,.$$

Thus $(X_1, X_2) = 0$ since $\lambda_1 \neq \lambda_2$.

3. If there is a repeated eigenvalue, the eigenvectors can still be orthogonalized using the Gram-Schmidt procedure. So without loss of generality, we may assume any two eigenvectors X_i and X_j are orthogonal to each other for $i \neq j$. Then if

$$f(x) = \sum_{n=1}^{\infty} A_n X_n\,;$$

taking the inner product with X_n we get

$$(f, X_n) = \sum_{m=1}^{\infty} (A_m X_m, X_n) = A_n(X_n, X_n)\,.$$

□

10 Convergence

So far all of our discussion about Fourier series has been formal, we haven't really touched upon when the function is equal to its fourier series and in what sense does the equality hold. One way to define equality is that if a generalized notion of "distance" between them is 0. For example in finite dimensions, say $\mathbb{C}^N$, there are several notions of distance between two vectors x and y, given by

$$\|x-y\|_{\ell^p} = (\sum_{n=1}^{N} |x_n - y_n|^p)^{\frac{1}{p}},$$

for any $p > 1$. It is straightforward to show that $\|\cdot\|_{\ell^p}$ satisfies all the properties of being a norm. Thus, we can say a sequence of vectors x_n converges to the vector x if

$$\lim_{n\to\infty} \|x_n - x\|_{\ell^p} = 0 .$$

Fortunately, in finite dimensions, all notions of convergence are equivalent, i.e. if x_n converges to x in the ℓ^p norm, then the sequence also converges to x in any other norm ℓ^q.

However, in infinite dimensions, things are not that straightforward. Let us look at three notions of convergence. Consider a sequence of functions $f_n(x)$ defined on the interval $[-1, 1]$.

Definition 10.1. *Pointwise convergence. f_n converges to f pointwise if*

$$\lim_{n\to\infty} |f_n(x) - f(x)| \to 0 \quad \forall x \in [-1, 1] .$$

Definition 10.2. *Uniform convergence. f_n converges to f uniformly if*

$$\lim_{n\to\infty} \sup_{x\in[-1,1]} |f_n(x) - f(x)| \to 0 .$$

Just like we have norms in finite dimensional spaces, we can define norms on function spaces too. The $\|\cdot\|_{\mathbb{L}^p}$ norm of a function on the interval $[-1, 1]$ is defined to be:

$$\|f\|_{\mathbb{L}^p[-1,1]} = \left(\int_{-1}^{1} |f(x)|^p \, dx\right)^{\frac{1}{p}} .$$

Exercise 10.1. *Hard exercise. Prove that $\|\cdot\|_{\mathbb{L}^p[-1,1]}$ defines a norm for $p \geq 1$.*

In particular, we will focus on the case $p = 2$.

Definition 10.3. *Convergence in $\mathbb{L}^2$. f_n converges to f in $\mathbb{L}^2$ if*

$$\lim_{n\to\infty} \|f_n - f\|_{\mathbb{L}^2[-1,1]} = \lim_{n\to\infty} \sqrt{\int_{-1}^{1} |f_n(x) - f(x)|^2 \, dx} = 0.$$

It is straightforward to show that if a sequence of functions f_n converges uniformly to the function f, then the sequence of functions converge pointwise and in $\mathbb{L}^2$.

$$\text{uniform convergence} \implies \text{point wise convergence}, \quad \text{uniform convergence} \implies \text{convergence in } \mathbb{L}^2 .$$

Consider the following two examples.

Example 10.1. *Let $f_n(x) = x^n$, then f_n converges pointwise to the function ' $f(x)$ on $[0, 1]$ where the $f(x) = 0$ for $0 \leq x < 1$ and $f(x) = 1$ for $x = 1$. However*

$$\max_{x\in[0,1]} |f_n(x) - f(x)| \geq \sup_{x\in[0,1)} |f_n(x) - f(x)| = \sup_{x\in[0,1)} x^n = 1 .$$

Thus, $f_n(x)$ does not uniformly converge to the function f.

The above example shows that pointwise convergence does not necessarily imply uniform convergence.

Example 10.2. *Let $f_n(x) = \sqrt{n}e^{-nx^2}$ defined on $(0, 1)$. Then $f_n(x)$ converges pointwise to the function 0 on $(0, 1)$. However,*

$$\int_0^1 |f_n(x) - f(x)|^2 = \int_0^1 ne^{-2nx^2} dx = \int_0^{\sqrt{n}} \sqrt{n}e^{-2y^2} dy \to \infty \quad as \quad n \to \infty .$$

Thus, a sequence of function that converges pointwise need not necessarily converge in $\mathbb{L}^2$.

Exercise 10.2. *Construct a counter example to prove that convergence in $\mathbb{L}^2$ does not imply point wise convergence.*

The purpose of the above discussion was to illustrate that one has to be careful and precise about the notion of convergence in the context of function spaces. Things are not as nice as in finite dimensions. However, the situation is not too grim for Fourier series of a function. Under "reasonable assumptions" on the smoothness of a function f, we can show that the fourier series of f converges uniformly to the function f.

10.1 Convergence in $\mathbb{L}^2$

Theorem 10.1. *The fourier series converges to $f(x)$ in the $\mathbb{L}^2$ sense if $f(x) \in \mathbb{L}^2[-1,1]$, i.e.*

$$\int_{-1}^{1} |f(x)|^2 \, dx < \infty \, .$$

It is important to note, that the Fourier series of f converges to f in the $\mathbb{L}^2$ sense for a large class of functions. The function f need not even be continuous. Proving convergence in $\mathbb{L}^2$ requires some additional machinery and is outside the scope of discussion for the class (there will be a problem in the next practice problem set). It can be shown that proving convergence in $\mathbb{L}^2$ of a fourier series is equivalent to *completeness* of the eigenfunction basis, i.e. e^{inx}. A basis of functions is a complete basis of $\mathbb{L}^2$, if every function in $\mathbb{L}^2$ can be arbitrarily well approximated by finite linear combinations of the basis of functions.

However, a much simpler result can be readily shown which we discuss next.

Lemma 10.1. *Bessel's inequality. Suppose the generalized fourier series of f is given by*

$$f(x) \sim \sum_{n=1}^{\infty} A_n X_n \, .$$

If $f(x) \in \mathbb{L}^2[-1,1]$, then

$$\sum_{n=-\infty}^{\infty} |A_n|^2 (X_n, X_n) \leq \int_{-1}^{1} |f(x)|^2 \, dx \, ,$$

Here we use $\sim$ instead of $=$, as we do not yet know if the fourier series of a function is equal to the function in any sense.

Proof.

$$\begin{aligned}
0 \leq \|f - \sum_{n=1}^{N} A_n X_n\|^2_{\mathbb{L}^2[-1,1]} &= \int_{-1}^{1} |f(x) - \sum_{n=1}^{N} A_n X_n|^2 \, dx \\
&= \int_{-1}^{1} |f(x)|^2 \, dx + \sum_{n=1}^{N} |A_n|^2 (X_n, X_n) - \sum_{n=1}^{N} 2A_n (f(x), X_n) \, , \\
&= \int_{-1}^{1} |f(x)|^2 \, dx + \sum_{n=1}^{N} |A_n|^2 (X_n, X_n) - \sum_{n=1}^{N} 2|A_n|^2 (X_n, X_n) \, , \quad \text{Since } A_n(X_n, X_n) = (f, X_n) \\
&= \int_{-1}^{1} |f(x)|^2 \, dx - \sum_{n=1}^{N} |A_n|^2 (X_n, X_n)
\end{aligned}$$

$$\therefore \sum_{n=1}^{N} |A_n|^2 (X_n, X_n) \leq \|f\|^2_{\mathbb{L}^2[-1,1]} \, .$$

□

10.2 Pointwise and uniform convergence

We now show that if f is sufficiently smooth, i.e. it has one continuous derivative and is periodic on $[-1,1]$, then the fourier series of f converges uniformly to $f(x)$. We show this in two steps, we first show that the fourier series of f converges pointwise to $f(x)$, and then upgrade the pointwise convergence to uniform convergence. We periodically extend $f(x)$ to a function on the whole real line, by the formula $f(x+2n) = f(x)$ for any $n \in \mathbb{N}$.

Let $S_N[f](x)$ denote the truncated fourier series of $f(x)$ given by

$$S_N[f](x) = \sum_{n=-N}^{N} c_n e^{i\pi n x} \, .$$

Then,

$$
\begin{aligned}
S_N[f](x) - f(x) &= \sum_{n=-N}^{N} \left(\frac{1}{2} \int_{-1}^{1} f(y) e^{-in\pi y}\, dy \right) e^{in\pi x} - f(x) \\
&= \sum_{n=-N}^{N} \frac{1}{2} \int_{-1}^{1} (f(y) - f(x)) e^{in\pi(x-y)}\, dy \quad (\text{Since } \int_{-1}^{1} e^{in\pi y} dy = 0 \text{ if } n \neq 0.) \\
&= \frac{1}{2} \int_{-1}^{1} (f(y) - f(x)) \sum_{n=-N}^{N} e^{-in\pi(x-y)}\, dy \\
&= \frac{1}{2} \int_{-1}^{1} (f(y) - f(x)) \frac{\sin\left((N+1/2)\pi(x-y)\right)}{\sin\left(\pi/2(x-y)\right)}\, dy \\
&= \frac{1}{2} \int_{-1}^{1} (f(x+z) - f(x)) K_N(z)\, dz, \quad (z = x - y \text{ and using the periodicity of } f \text{ and } K_N)
\end{aligned}
$$

where $K_N(z)$ is the Dirichlet kernel given by

$$
K_N(z) = \frac{\sin\left(N+1/2\right)\pi z}{\sin\left((\pi/2) z\right)}\,.
$$

To complete the proof, we appeal to yet another lemma.

Lemma 10.2. *Riemann Lesbegue. Suppose $f(x)$ is a continuous periodic function with period 2, then*

$$
\lim_{n\to\infty} \int_{-1}^{1} f(x)\sin(n\pi x)\, dx = \lim_{n\to\infty} \int_{-1}^{1} f(x)\cos(n\pi x)\, dx = \lim_{n\to\infty} \int_{-1}^{1} f(x) e^{in\pi x}\, dx = \lim_{n\to\infty} \int_{-1}^{1} f(x) e^{-in\pi x}\, dx = 0
$$

Proof. We show the result for the sine case and the rest of the results follow in a similar manner. Let

$$
I_n = \int_{-1}^{1} f(x)\sin(n\pi x)\, dx\,.
$$

$$
\int_{-1}^{1} f(x)\sin(n\pi(x + \frac{1}{n})\, dx = \int_{-1}^{1} f(x)\sin((n\pi x) + \pi)\, dx = -I_n\,.
$$

Thus,

$$
\begin{aligned}
2I_n &= \int_{-1}^{1} \sin(n\pi x) f(x)\, dx - \int_{-1}^{1} \sin(n\pi(x + \frac{1}{n}) f(x)\, dx \\
&= \int_{-1}^{1} \sin(n\pi x) f(x)\, dx - \int_{-1}^{1} \sin(n\pi x) f(x - \frac{1}{n}), \quad (x \to x - \frac{1}{n} \text{ and using periodicity of } f \text{ and } \sin) \\
&= \int_{-1}^{1} \sin(n\pi x)(f(x) - f(x - \frac{1}{n})\,.
\end{aligned}
$$

Exercise 10.3. *Show that $\sin(n\pi x)(f(x) - f(x - \frac{1}{n})$ uniformly converges to 0 for $x \in [-1, 1]$. (hint: pointwise convergence $\Longrightarrow$ uniform convergence on compact intervals and sandwich theorem is also your friend)*

Since the integrand converges uniformly to 0,

$$
\lim_{n\to\infty} I_n = \frac{1}{2} \lim_{n\to\infty} \int_{-1}^{1} \sin(n\pi x)(f(x) - f(x - \frac{1}{n})\, dx = \frac{1}{2} \int_{-1}^{1} \lim_{n\to\infty} \sin(n\pi x)(f(x) - f(x - \frac{1}{n}))\, dx = 0\,.
$$

□

Theorem 10.2. *If f is periodic and has one continuous derivative then $S_N[f](x) \to f(x)$ pointwise.*

Proof.

$$
\begin{aligned}
S_N[f](x) - f(x) &= \int_{-1}^{1} (\frac{f(x+z) - f(x)}{\sin(\pi/2) z}) \sin((N+1/2)\pi z)\, dz \\
&= \int_{-1}^{1} (\frac{f(x+z) - f(x)}{\sin(\pi/2) z}) \cdot \cos(\pi/2) z \cdot \sin N\pi z\, dz + \int_{-1}^{1} (f(x+z) - f(x)) \cos N\pi z\, dz
\end{aligned}
$$

If $f \in C^1$ i.e. f has one continuous derivative, then using L'Hopital's rule, it is striaghtforward to show that the functions

$$(f(x+z) - f(x)) \cdot \cot(\pi/2)z \quad \text{and} \quad (f(x+z) - f(x))\,,$$

are continuous. The result of pointwise convergence then follows from the Riemann Lesbegue lemma. □

Theorem 10.3. *If f is periodic and has one continuous derivative then $S_N[f](x) \to f(x)$ uniformly.*

Proof. We already know that the fourier series of f converges to f pointwise. Let

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x}\,.$$

Then

$$\begin{aligned} c_n &= \frac{1}{2}\int -11 f(x) e^{-in\pi x}\, dx \\ &= -\frac{1}{2in\pi}\int_{-1}^{1} f'(x) e^{-in\pi x}\, dx \quad \text{Integrating by parts and } f \text{ is periodic} \\ &= -\frac{1}{2in\pi} d_n\,, \end{aligned}$$

where

$$f'(x) \sim \sum_{n=-\infty}^{\infty} d_n e^{in\pi x}\,,$$

i.e. d_n are the fourier coefficients of $f'(x)$.

$$\begin{aligned} |f(x) - S_N[f](x)| &= |\sum_{|n|>N} c_n e^{in\pi x}| \\ &= |\sum_{|n|>N} \frac{d_n}{2i\pi n} e^{in\pi x}| \\ &\le \left(\sum_{|n|>N} \frac{1}{4\pi^2 n^2}\right)^{\frac{1}{2}} \left(\sum_{|n|>N} |d_n e^{in\pi x}|^2\right)^{\frac{1}{2}} \quad \left(\sum_n a_n b_n \le (\sum_n |a_n|^2)^{\frac{1}{2}} (\sum_n |b_n|^2)^{\frac{1}{2}}\right) \\ &= \left(\sum_{|n|>N} \frac{1}{4\pi^2 n^2}\right)^{\frac{1}{2}} \left(\sum_{|n|>N} |d_n|^2\right)^{\frac{1}{2}} \end{aligned}$$

$$\therefore \lim_{N\to\infty} \sup_{x\in[-1,1]} |f(x) - S_N[f](x)| \le \lim_{N\to\infty} \left(\sum_{|n|>N} \frac{1}{4\pi^2 n^2}\right)^{\frac{1}{2}} \left(\sum_{|n|>N} |d_n|^2\right)^{\frac{1}{2}} = 0$$

The last limit is zero since $\frac{1}{n^2}$ and $|d_n|^2$ are convergent and the tail of a convergent series goes to zero ($|d_n|^2$ is a convergent series due to Bessel's inequality applied to the fourier coefficients of $f'(x)$). □

10.3 Derivatives of series of functions

In the previous sections, we have obtained solutions to the dirichlet problem for the wave and heat equations as

$$u_w(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{\ell}\right) + B_n \sin\left(\frac{n\pi ct}{\ell}\right)\right) \sin\left(\frac{n\pi x}{\ell}\right), \tag{80}$$

and

$$u_d(x,t) = \sum_{n=1}^{\infty} A_n \exp\left(-\frac{n^2\pi^2 kt}{\ell^2}\right) \sin\left(\frac{n\pi x}{\ell}\right). \tag{81}$$

To verify that these expressions indeed satisfy the PDE and the right initial conditions, we need the following additional result regarding the derivative of a series of functions. Let

$$f(x) = \sum_{n=1}^{\infty} f_n(x)\,.$$

Suppose the above series converges uniformly, $f_n'(x)$ exists for all n and furthermore the series

$$\sum_{n=1}^{\infty} f_n'(x)$$

converges uniformly, then

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x)\,.$$

Theorem 10.4. *If $\phi \in C^3$, $\psi \in C^2$ and ϕ, ψ are periodic, then equation* (80) *satisfies the wave equation with homogeneous dirichlet boundary conditions.*

Proof. If $\phi \in C^3$, i.e. if ϕ has three continuous derivatives, then the sine series of ϕ,

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{\ell}\right),$$

converges uniformly to ϕ. Furthermore,

$$\begin{aligned}
A_n &= \frac{2}{\ell}\int_0^\ell \phi(x)\sin\left(\frac{n\pi x}{\ell}\right)dx \\
&= \frac{2}{n\pi}\int_0^\ell \phi'(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \quad \text{Integration by parts and periodicity of } \phi \\
&= -\frac{2\ell}{n^2\pi^2}\int_0^\ell \phi''(x)\sin\left(\frac{n\pi x}{\ell}\right)dx \\
&= -\frac{2\ell^2}{n^3\pi^3}\int_0^\ell \phi'''(x)\cos\left(\frac{n\pi x}{\ell}\right)dx \\
&= -\frac{\ell^3}{n^3\pi^3}\alpha_n\,,
\end{aligned}$$

where α_n are the coefficients of the cosine series of $\phi'''(x)$ Let

$$\psi(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{\ell} B_n \sin(\frac{n\pi x}{\ell})\,.$$

Then proceeding as above and integrating by parts twice, we get

$$B_n = -\frac{\ell^3}{n^3\pi^3 c}\beta_n\,,$$

where β_n are the coefficients of the sine series of $\psi''(x)$. Thus,

$$u(x,t) = -\sum_{n=1}^{\infty}\left(\frac{\alpha_n\ell^3}{n^3\pi^3}\cos\left(\frac{n\pi ct}{\ell}\right) + \frac{\beta_n\ell^3}{n^3\pi^3 c}\sin\left(\frac{n\pi ct}{\ell}\right)\right)\sin\left(\frac{n\pi x}{\ell}\right)$$

where owing to Bessel's inequality applied to $\phi'''(x)$ and $\psi''(x)$, the coefficients α_n and β_n satisfy

$$\sum_n |\alpha_n|^2 < \infty\,, \quad \sum_n |\beta_n|^2 < \infty\,.$$

In order to show that

$$\partial_x u(x,t) = -\sum_{n=1}^{\infty}\left(\frac{\alpha_n\ell^2}{n^2\pi^2}\cos\left(\frac{n\pi ct}{\ell}\right) + \frac{\beta_n\ell^2}{n^2\pi^2 c}\sin\left(\frac{n\pi ct}{\ell}\right)\right)\cos\left(\frac{n\pi x}{\ell}\right),$$

we just need to show that the series

$$-\sum_{n=1}^{\infty}\left(\frac{\alpha_n\ell^3}{n^3\pi^3}\cos\left(\frac{n\pi ct}{\ell}\right) + \frac{\beta_n\ell^3}{n^3\pi^3 c}\sin\left(\frac{n\pi ct}{\ell}\right)\right)\sin\left(\frac{n\pi x}{\ell}\right),$$

and

$$-\sum_{n=1}^{\infty}\left(\frac{\alpha_n\ell^2}{n^2\pi^2}\cos\left(\frac{n\pi ct}{\ell}\right) + \frac{\beta_n\ell^2}{n^2\pi^2 c}\sin\left(\frac{n\pi ct}{\ell}\right)\right)\cos\left(\frac{n\pi x}{\ell}\right),$$

converge uniformly on $[-1,1]$. We first note that

$$\left|\left(\frac{\alpha_n \ell^3}{n^3\pi^3}\cos\left(\frac{n\pi ct}{\ell}\right)+\frac{\beta_n\ell^3}{n^3\pi^3 c}\sin\left(\frac{n\pi ct}{\ell}\right)\right)\sin\left(\frac{n\pi x}{\ell}\right)\right| \leq \frac{C}{n^3} \quad \text{(For some constant } C\text{)}\,,$$

since α_n and β_n are bounded. To show uniform convergence, we need to show that

$$\lim_{N\to\infty} I_N = \lim_{N\to\infty}\sup_{x\in[0,\ell]}\left|\sum_{n>N}\left(\frac{\alpha_n \ell^3}{n^3\pi^3}\cos\left(\frac{n\pi ct}{\ell}\right)+\frac{\beta_n\ell^3}{n^3\pi^3 c}\sin\left(\frac{n\pi ct}{\ell}\right)\right)\sin\left(\frac{n\pi x}{\ell}\right)\right| = 0$$

Using the inequality above

$$I_N \leq \sup_{x\in[0,\ell]}\left|\sum_{n>N}\frac{C}{n^3}\right|$$

Thus,

$$\lim_{N\to\infty} I_N = 0\,,$$

since $\sum_{n>0}\frac{1}{n^3}$ is a convergent series. The proof of uniform convergence of the second series follows in a similar fashion. We can proceed in a similar fashion to show that

$$\begin{aligned}
\partial_{xx}u_w(x,t) &= \sum_{n=1}^{\infty}\partial_{xx}u_n(x,t)\\
\partial_{tt}u_w(x,t) &= \sum_{n=1}^{\infty}\partial_{tt}u_n(x,t)\\
\left(\partial_{tt}-c^2\partial_{xx}\right)u_w(x,t) &= \sum_{n=1}^{\infty}\left(\partial_{tt}-c^2\partial_{xx}\right)u_n(x,t) = 0\,,
\end{aligned}$$

since $u_n(x,t)$ satisfies the wave equation for each n. □

Theorem 10.5. *If $\phi \in C^1$, then equation* (81) *satisfies the diffusion equation with homogeneous dirichlet boundary conditions.*

Exercise 10.4. *Prove therorem 10.5*

11 Harmonic functions

We now proceed to studying partial differential equations in higher spatial dimensions and begin with Laplace's equation:

Laplace's equation is two and three dimensions is given by

$$\begin{aligned}
\nabla\cdot\nabla u = \Delta u &= u_{xx}+u_{yy} = 0 \quad \text{in 2D}\,, &(82)\\
&= u_{xx}+u_{yy}+u_{zz} = 0 \quad \text{in 3D}\,. &(83)
\end{aligned}$$

Solutions to Laplace's equation in two and three dimensions are known as *harmonic functions*. The inhomogeneous version of Laplace's equation

$$\Delta u = f\,,$$

is called *Poisson's equation*. This equation has been extensively studied by mathematicians and physicsts in several contexts as it has applications in and connections to electro/magnetostatics, potential flow, complex analysis, and brownian motion.

In one spatial dimension, solutions to Laplace equation are not too exciting. The solution to

$$u_{xx} = f\,, \quad u(0) = c_1\,, \quad u(1) = c_2\,,$$

is given by

$$u(x) = \int^x\int^s f(t)dtds + A + Bx\,,$$

where A and B can be determined by solving a linear system.

11.1 Preliminaries

Let D denote a bounded, open set in $\mathbb{R}^{2,3}$. Points in two or three dimensions will be denoted by bold face symbols, i.e $\boldsymbol{x} = (x,y)$ or $\boldsymbol{x} = (x,y,z)$ where x,y,z are the coordinates of the vector. Let ∂D denote the boundary of D. Let $\boldsymbol{n}(\boldsymbol{x})$ denote the outward normal at $\boldsymbol{x}\in\partial D$.

11.2 Boundary conditions

The *Dirichlet problem* for the Poisson problem is given by

$$\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \boldsymbol{x} \in D\,, \qquad u(\boldsymbol{x}) = g(\boldsymbol{x}) \quad \boldsymbol{x} \in \partial D\,.$$

The *Neumann problem* for the Poisson problem is given by

$$\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \boldsymbol{x} \in D\,, \qquad \frac{\partial u}{\partial n}(\boldsymbol{x}) = g(\boldsymbol{x}) \quad \boldsymbol{x} \in \partial D\,.$$

The *Robin problem* for the Poisson problem is given by

$$\Delta u(\boldsymbol{x}) = f(\boldsymbol{x}) \quad \boldsymbol{x} \in D\,, \qquad \frac{\partial u}{\partial n}(\boldsymbol{x}) + au(\boldsymbol{x}) = g(\boldsymbol{x}) \quad \boldsymbol{x} \in \partial D\,.$$

11.3 Maximum principle

Analogous to the diffusion equation in $1D$, harmonic functions also satisfy a maximum principle. Harmonic functions achieve their maximum and minimum values on the boundary of the domain D. One of the advantages of maximum principles is that uniqueness of solutions to the PDE is free if the PDE satisfies a maximum principle. Usually PDEs which have a maximum principle also result in smooth solutions to the PDE.

Theorem 11.1. *Weak maximum principle Suppose D is an open, bounded and conneted set in $\mathbb{R}^{2,3}$. Let $\overline{D} = D \cup \partial D$ denote the closure of the domain D Suppose u is a harmonic function in D and continuous in $\overline{D}$, then*

$$\max_{\boldsymbol{x}\in\overline{D}} u(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\partial D} u(\boldsymbol{x})\,,$$

i.e. u achieves it's maximum value on the boundary.

There exists a stronger version of the maximum principle, which goes on to further state that a harmonic function u cannot achieve it's maximum value in the interior D unless it is a constant. We postpone the discussion of that result for a later subsection.

Proof. We know that for a function of one variable $f(x)$ which achieves it's maximum value at x_0 satisfies $f'(x_0) = 0$ and $f''(x_0) \leq 0$. Suppose u achieves it's maximum value at $\boldsymbol{x}_0$ in the interior, then applying the above principle along the x axis and y axis centered at $\boldsymbol{x}_0$, we get that $\partial_{xx} u(\boldsymbol{x}_0) \leq 0$, and $\partial_{yy} u(\boldsymbol{x}_0) \leq 0$. However, since u is harmonic, it also satisfies $(\partial_{xx} + \partial_{yy})u(\boldsymbol{x}_0) = 0$. Thus, we are ϵ away from a contradiction and proving our result. To get the extra ϵ, let $\epsilon > 0$ and consider

$$v(\boldsymbol{x}) = u(\boldsymbol{x}) + \epsilon|\boldsymbol{x}|^2\,.$$

Note that,

$$\Delta|\boldsymbol{x}|^2 = (\partial_{xx} + \partial_{yy})(x^2 + y^2) = 2 + 2 = 4\,.$$

The reason for the choice of the above function is that, if u is harmonic

$$\Delta v = \Delta u + \epsilon\Delta|\boldsymbol{x}|^2 = 0 + 4\epsilon > 0\,.$$

Using the argument above for the function v, if $\boldsymbol{x}_0 \in D$ is the location of the maximum then, $\Delta v(\boldsymbol{x}_0) \leq 0$, which contradicts $\Delta v(\boldsymbol{x}_0) = 4\epsilon > 0$. Thus, v achieves it's maximum value on the boundary, i.e.

$$v(\boldsymbol{x}) \leq \max_{\boldsymbol{x}\in\overline{D}} v(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\partial D} v(\boldsymbol{x})\,.$$

Also,

$$u(\boldsymbol{x}) = v(\boldsymbol{x}) - \epsilon|\boldsymbol{x}|^2 < v(\boldsymbol{x})\,.$$

Furthermore, since the boundary ∂D is compact, there exists $\boldsymbol{x}_0 \in \partial D$ such that $v(\boldsymbol{x}_0) = \max_{\boldsymbol{x}\in\partial D} v(\boldsymbol{x})$. Combining everything we have, we get

$$u(\boldsymbol{x}) < v(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\partial D} v(\boldsymbol{x}) = v(\boldsymbol{x}_0) = u(\boldsymbol{x}_0) + \epsilon|\boldsymbol{x}_0|^2\,,$$

for all $\boldsymbol{x} \in \overline{D}$. Continuing further, since the domain D is bounded, $\exists R_0$ such that $|\boldsymbol{x}| \leq R_0$ for all $\boldsymbol{x} \in \overline{D}$. Furthermore, $u(\boldsymbol{x}_0) \leq \max_{\boldsymbol{x}\in\partial D} u(\boldsymbol{x})$. Thus,

$$u(\boldsymbol{x}) < u(\boldsymbol{x}_0) + \epsilon|\boldsymbol{x}_0|^2 \leq \max_{\boldsymbol{x}\in\partial D} u(\boldsymbol{x}) + \epsilon R_0^2\,.$$

Taking the limit as ϵ going to 0, we get

$$\max_{\boldsymbol{x}\in\overline{D}} u(\boldsymbol{x}) \leq \max_{\boldsymbol{x}\in\partial D} u(\boldsymbol{x})\,.$$

Furthermore, since $\partial D \subseteq \overline{D}$, it is straightforward to show that

$$\max_{\boldsymbol{x}\in\partial D} u(\boldsymbol{x}) \le \max_{\boldsymbol{x}\in\overline{D}} u(\boldsymbol{x}) .$$

Combining these two results, we get

$$\max_{\boldsymbol{x}\in\overline{D}} u(\boldsymbol{x}) = \max_{\boldsymbol{x}\in\partial D} u(\boldsymbol{x}) .$$

□

11.4 Uniqueness to the Dirichlet problem at Poisson's problem

Using the maximum principle, it is straightforward to prove unqiueness of solutions to the Poisson problem with Dirichlet boundary conditions. Suppose u_1 and u_2 satisfy the Poisson problem:

$$\Delta u_1 = f \quad \boldsymbol{x}\in D \qquad u_1 = g \quad \boldsymbol{x}\in\partial D ,$$

$$\Delta u_2 = f \quad \boldsymbol{x}\in D \qquad u_2 = g \quad \boldsymbol{x}\in\partial D .$$

Then the difference $w = u_1 - u_2$ satisfies

$$\Delta w = \Delta u_1 - \Delta u_2 = f - f = 0 , \quad \boldsymbol{x}\in D \qquad w = u_1 - u_2 = g - g = 0 , \quad \boldsymbol{x}\in\partial D .$$

Since, w is harmonic, w satisfies the maximum and minimum principle. Furthermore, $w = 0$ on the boundary, so $\max w = \min w = 0$. Using the maximum principle, we conclude that

$$0 = \min_{\boldsymbol{x}\in\partial D} w = \min_{\boldsymbol{x}\in\overline{D}} w \le w(\boldsymbol{x}) \le \max_{\boldsymbol{x}\in\overline{D}} w = \max_{\boldsymbol{x}\in\partial D} w = 0$$

11.5 Separation of variables

We now turn to computing harmonic functions on simple domains such as rectangles, cubes and disks where the solutions can be obtained using separation of variables.

11.5.1 Two dimensions

Consider the Dirichlet problem for Laplace's equation on the rectangle R whose vertices are given by $(0,0), (\pi,0), (\pi,1), (0,1)$:

$$\Delta u = 0 , \quad \boldsymbol{x}\in R , \qquad u(x,0) = h_1(x) \quad y = 0 , \qquad u(x,1) = h_2(x) \quad y = 1 , \qquad u(0,y) = g_1(y) \quad x = 0 , \qquad u(\pi,y) = g_2(y) \quad x = \pi .$$

Using linearity, we can decompose the above problem into 4 simpler problems:

$$u = u_1 + u_2 + u_3 + u_4 ,$$

where

$$\Delta u_1 = 0 , \quad \boldsymbol{x}\in R , \qquad u_1(x,0) = h_1(x) \quad y = 0 , \qquad u_1(x,1) = 0 \quad y = 1 , \qquad u_1(0,y) = 0 \quad x = 0 , \qquad u_1(\pi,y) = 0 \quad x = \pi .$$

$$\Delta u_2 = 0 , \quad \boldsymbol{x}\in R , \qquad u_2(x,0) = 0 \quad y = 0 , \qquad u_2(x,1) = h_2(x) \quad y = 1 , \qquad u_2(0,y) = 0 \quad x = 0 , \qquad u_2(\pi,y) = 0 \quad x = \pi .$$

$$\Delta u_3 = 0 , \quad \boldsymbol{x}\in R , \qquad u_3(x,0) = 0 \quad y = 0 , \qquad u_3(x,1) = 0 \quad y = 1 , \qquad u_3(0,y) = g_1(y) \quad x = 0 , \qquad u_3(\pi,y) = 0 \quad x = \pi .$$

$$\Delta u_4 = 0 , \quad \boldsymbol{x}\in R , \qquad u_4(x,0) = 0 \quad y = 0 , \qquad u_4(x,1) = 0 \quad y = 1 , \qquad u_4(0,y) = 0 \quad x = 0 , \qquad u_4(\pi,y) = g_2(y) \quad x = \pi .$$

Let us focus on the computation of one of these functions, $u_2(\boldsymbol{x})$. We will compute $u_1(\boldsymbol{x})$ using separation of variables. We look for solutions of the form

$$u_2(\boldsymbol{x}) = X(x)Y(y) .$$

On imposing the boundary conditions, we obtain the following boundary conditions for $X(x)$ and $Y(y)$.

$$y = 0 \quad \text{edge} \qquad X(x)Y(0) = 0 \implies Y(0) = 0 ,$$

$$y = 1 \quad \text{edge} \qquad X(x)Y(1) = h_2(x) \implies X(x) = h_2(x)/Y_1,$$

$$x = 0 \quad \text{edge} \qquad X(0)Y(y) = 0 \implies X(0) = 0 ,$$

$$x = \pi \quad \text{edge} \qquad X(\pi)Y(y) = 0 \implies X(\pi) = 0 ,$$

Plugging in the ansatz for u in the PDE, we get

$$X''Y + Y''X = 0 \implies \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$$

The reason for the choice of $-\lambda^2$ as opposed to λ^2 is due to the fact that $X(x)$ satisfies the boundary conditions $X(0) = 0 = X(\pi)$. We know that the second derivative operator with those boundary conditions is a negative definite operator. The eigenvalues of the operator and the corresponding eigenvectors for the above operator are

$$-\lambda_n^2 = -n^2 \qquad X_n(x) = \sin(nx)\,.$$

The corresponding differential equation for $Y(y)$ is

$$Y_n(y)'' = \lambda_n^2 Y(y) \implies Y_n(y) = A\cosh(ny) + B\sinh(ny)\,.$$

On imposing the boundary condition $Y(0) = 0$, we get

$$Y_n(y) = \sinh(ny)\,.$$

Thus, the separation of variable solutions for $u_2(\boldsymbol{x})$ are given by

$$X_n(x)Y_n(y) = \sin(nx)\sinh(ny)\,.$$

Using linearity of the problem, any linear combination of the separation of variables solutions is also to the problem. Thus,

$$u_2(\boldsymbol{x}) = \sum_{n=1}^{\infty} A_n \sin(nx)\sinh(ny)\,.$$

The coefficients A_n can be determined by imposing the last boundary condition $u_2(x,1) = h_2(x)$.

$$h_2(x) = \sum_{n=1}^{\infty} A_n \sinh(n)\sin(nx)\,.$$

Expressing $h_2(x)$ as a sine series, we get

$$A_n \sinh(n) = \frac{2}{\pi}\int_0^{\pi} h_2(s)\sin(ns)ds\,,$$

which completes the computation of $u_2(\boldsymbol{x})$.

11.5.2 Three dimensions

We now turn our attention to three dimensions to demonstrate that the same idea works out. Consider the Dirichlet problem for Laplace's equation on the cube R with side length π. As before, we can split the problem into simpler boundary value problem given by

$$\Delta u = 0\,, \quad \boldsymbol{x} \in R = \{0 < x < \pi, 0 < y < \pi, 0 < z < \pi\}\,,$$
$$u(0,y,z) = 0\,, \quad u(\pi,y,z) = g(y,z)\,, \quad u(x,0,z) = 0\,, \quad u(x,\pi,z) = 0\,, \quad u(x,y,0) = 0\,, \quad u(x,y,\pi) = 0\,.$$

We look for solutions of the form

$$u(\boldsymbol{x}) = X(x)Y(y)Z(z)\,.$$

On imposing the boundary conditions, we obtain the following boundary conditions for $X(x)$ and $Y(y)$.

$$\begin{aligned}
x = 0 \quad &\text{face} \quad X(0)Y(y)Z(z) = 0 \implies X(0) = 0\,,\\
x = \pi \quad &\text{face} \quad X(\pi)Y(y)Z(z) = g(y,z) \implies Y(y)Z(z) = \frac{g(y,z)}{X(\pi)}\,,\\
y = 0 \quad &\text{face} \quad X(x)Y(0)Z(z) = 0 \implies Y(0) = 0\,,\\
y = \pi \quad &\text{face} \quad X(x)Y(\pi)Z(z) = 0 \implies Y(\pi) = 0\,,\\
z = 0 \quad &\text{face} \quad X(x)Y(y)Z(0) = 0 \implies Z(0) = 0\,,\\
z = \pi \quad &\text{face} \quad X(x)Y(y)Z(\pi) = 0 \implies Z(\pi) = 0\,.
\end{aligned}$$

Plugging in the ansatz for u in the PDE, we get

$$X''YZ + Y''XZ + Z''XY = 0 \implies \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0\,.$$

The only way the above equation would be satisfied if

$$\frac{Y''}{Y} = -m^2\,, \quad \frac{Z''}{Z} = -n^2\,, \quad \frac{X''}{X} = m^2 + n^2\,.$$

Again the choice for the signs for the constants is due to the boundary conditions $Y(0) = 0 = Y(\pi)$ and $Z(0) = 0 = Z(\pi)$. The eigenfunctions for $Y(y)$ and $Z(z)$ are

$$Y(y) = \sin(my)\,, \quad Z(z) = \sin(nz)\,.$$

The corresponding solution $X(x)$ is given by

$$X(x) = A\cosh(\sqrt{m^2+n^2}x) + B\sinh(\sqrt{m^2+n^2}x)\,.$$

On imposing the boundary condition $X(0) = 0$, we get

$$X(x) = A\sinh(\sqrt{m^2+n^2}x)\,,$$

and the final separation of variables solution is given by

$$u(x,y,z) = \sinh(\sqrt{m^2+n^2}x)\sin(my)\sin(nz)\,.$$

By linearity, any linear combination also satisfies the PDE

$$u(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin(my)\sin(nz)\sinh(\sqrt{m^2+n^2}x)\,,$$

where the coefficients A_{mn} can be obtained by imposing the boundary conditions $u(\pi,y,z) = g(y,z)$.

$$g(y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\beta_{mn}\sin(my)\sin(nz)\,,$$

where $\beta_{mn} = \sinh(\sqrt{m^2+n^2}\pi)$. This is the $2D$ analogue of the one dimension Fourier sine series. Fortunately, the extension to higher dimension is straightforward as the basis itself in a separation of variable form and moreover the basis elements $\sin(my)\sin(nz)$ are orthogonal. The inner product in this setup to measure orthogonality is

$$(f(y,z), g(y,z)) = \int_0^{\pi}\int_0^{\pi} f(y,z)g(y,z)\,dydz\,.$$

Consider two basis elements $\sin(m_1 y)\sin(n_1 z)$ and $\sin(m_2 y)\sin(n_2 z)$ where $(m_1,n_1) \neq (m_2,n_2)$, i.e. either $m_1 \neq m_2$ or $n_1 \neq n_2$ or both. For simplicity consider $m_1 \neq m_2$. Then the inner product between these two elements is given by

$$\int_0^{\pi}\int_0^{\pi} \sin(m_1 y)\sin(n_1 z)\sin(m_2 y)\sin(n_2 z)\,dydz = \int_0^{\pi}\sin(n_1 z)\sin(n_2 z)dz\int_0^{\pi}\sin(m_1 y)\sin(m_2 y)dy = 0\,,$$

since $m_1 \neq m_2 \implies$

$$\int_0^{\pi}\sin(m_1 y)\sin(m_2 y)dy = 0\,.$$

Thus, the coefficients $A_{mn}\beta_{mn}$ can be obtained by the simple formula

$$A_{mn}\beta_{mn} = \frac{4}{\pi^2}\int_0^{\pi}\int_0^{\pi} g(y,z)\sin(my)\sin(nz)\,dydz\,.$$

11.6 Poisson's formula - Disk in 2D

Another simple domain in two dimensions which lends itself to obtaining explicit solutions is the disk $x^2+y^2 \leq a^2$. Consider the dirichlet boundary value problem in two dimensions:

$$u_{xx}+u_{yy}=0 \quad x^2+y^2<a^2\,, \quad u=h(\theta) \quad x^2+y^2=a^2\,.$$

The domain can be thought about as a rectangle in polar coordinates (r,θ) where $0 \leq r \leq a$ and $\theta \in [0,2\pi)$, and thus we can look for separation of solutions of the form

$$u(r,\theta)=R(r)T(\theta)\,.$$

The PDE in polar coordinates can be rewritten as

$$\Delta u = u_{xx}+u_{yy}=u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}\,.$$

Plugging in the ansatz for u, we get

$$R''T+\frac{1}{r}R'T+\frac{1}{r^2}T''=0\,.$$

Dividing by u, we get

$$r^2\frac{R''}{R}+r\frac{R'}{R}=-\frac{T''}{T}\,.$$

Since the quantityu on the left is purely a function of r and the quantity on the right is purely a function θ, we conclude that both of them must be a constant. The natural boundary conditions for $T(\theta)$ are periodic boundary conditions, i.e $T(0)=T(2\pi)$ and $T'(0)=T'(2\pi)$ as $(r,0)$ and $(r,2\pi)$ are physically the same point in space. The eigenvalues and eigenfunctions corresponding to these boundary conditions are given by

$$-\frac{T_n''}{T_n}=n^2\,, \quad T_n(\theta)=A_n\cos(n\theta)+B_n\sin(n\theta)\,.$$

The equation for the radial function $R_n(r)$ assciated with T_n is

$$r^2R_n''+rR_n'-n^2R_n=0\,.$$

This is an Euler differential equation. Plugging in the ansatz $R_n=r^\alpha$, we get

$$\alpha(\alpha-1)r^2r^{\alpha-2}+\alpha r r^{\alpha-1}-n^2r^\alpha=0=r^\alpha(\alpha^2-n^2)\,.$$

Thus the two solutions linearly independent solutions for the radial function $R_n(r)$ are r^n and r^{-n}. However, as it seems, we just have one boundary condition for the radial function at $r=a$. The natural boundary condition at $r=0$ should be that the solution $R(r)T(\theta)$ should be *regular* at $r=0$ and hence we discard the solution r^{-n}. Thus, the separation of variables solutions for the PDE are given by

$$u(r,\theta)=r^n(A_n\cos(n\theta)+B_n\sin(n\theta))\,.$$

As always, a linear combination of all of these solutions is still a solution and we obtain the following general expression for the harmonic function u

$$u(r,\theta)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}r^n(A_n\cos(n\theta)+B_n\sin(n\theta))\,.$$

On imposing the boundary condition, we get

$$u(a,\theta)=h(\theta)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}r^n(A_n\cos(n\theta)+B_n\sin(n\theta))\,.$$

The Fourier coefficients A_n and B_n are given by

$$A_n=\frac{1}{a^n\pi}\int_0^{2\pi}h(\phi)\cos(n\phi)d\phi\,, \quad \frac{1}{a^n\pi}\int_0^{2\pi}h(\phi)\sin(n\phi)d\phi\,.$$

Plugging it back into the expression for u, we get

$$u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}h(\phi)d\phi+\sum_{n=1}^{\infty}\frac{2}{2\pi}\left(\frac{r}{a}\right)^n\int_0^{2\pi}\left(\cos(n\theta)\cos(n\phi)+\sin(n\theta)\sin(n\phi)\right)h(\phi)d\phi\,.$$

$$u(r,\theta)=\frac{1}{2\pi}\left(\int_0^{2\pi}h(\phi)d\phi+\int_0^{2\pi}\left(\sum_{n=1}^{\infty}2\left(\frac{r}{a}\right)^n\cos n(\theta-\phi)\right)h(\phi)d\phi\right)\,.$$

$$u(r,\theta) = \frac{1}{2\pi}\left(\int_0^{2\pi} h(\phi)d\phi + \int_0^{2\pi}\left(\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left(e^{in(\theta-\phi)} + e^{-in(\theta-\phi)}\right)\right)h(\phi)d\phi\right).$$

The above geometric series can be computed explicitly, and we get the following formula for the solution

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi}\int_0^{2\pi}\frac{h(\phi)}{a^2 - 2ar\cos(\theta-\phi) + r^2}d\phi$$

Let $\boldsymbol{x}' = (a,\phi)$ and $\boldsymbol{x} = (r,\theta)$ denote the polar coordinates of $\boldsymbol{x}$ and $\boldsymbol{x}'$. The above expression can be rewritten as

$$u(\boldsymbol{x}) = \frac{a^2 - |\boldsymbol{x}|^2}{2\pi a}\int_{|\boldsymbol{x}'|=a}\frac{u(\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|^2}ds'$$

Both of these expressions for the solution of Laplace's equation in the disk is often referred to as *Poisson's formula* (not to be confused with Poisson's equation $\Delta u = f$.)

11.6.1 Mean value property and the strong maximum principle

An interesting consequence of Poisson's formula is the mean value principle for harmonic functions stated below. Let $B_r(\boldsymbol{x}_0)$ denote the ball of radius r centered at $\boldsymbol{x}_0$:

$$B_r(\boldsymbol{x}_0)\{\boldsymbol{x} : |\boldsymbol{x}-\boldsymbol{x}_0| < r\},$$

and as always let $\partial B_r(\boldsymbol{x}_0)$ denote the boundary on $B_r(\boldsymbol{x}_0)$ which in this case is

$$\partial B_r(\boldsymbol{x}_0) = \{\boldsymbol{x} : |\boldsymbol{x}-\boldsymbol{x}_0| = r\}.$$

Theorem 11.2. *Suppose u is a harmonic function in D and let $\boldsymbol{x}_0 \in D$ and a sufficiently small such that $B_r(\boldsymbol{x}_0) \subset D$, then*

$$u(\boldsymbol{x}_0) = \frac{1}{2\pi a}\int_{\partial B_r(\boldsymbol{x}_0)} u(\boldsymbol{x}')dS = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + a\cos(\theta), y_0 + a\sin(\theta))\,d\theta\,.$$

Proof. This follows from a straight forward application of Poisson's formula on the disc. Let us consider the restriction of the harmonic function u to the disc $B_a(\boldsymbol{x}_0)$. The boundary data for the harmonic function is given by

$$h(\theta) = u(x_0 + a\cos(\theta), y_0 + a\sin(\theta))\,.$$

Then applying Poisson's formula and setting $r = 0$, we get

$$u(\boldsymbol{x}_0) = \frac{a^2 - 0^2}{2\pi}\int_0^{2\pi}\frac{h(\phi)}{a^2 - 2a\cdot 0\cos(\theta-\phi) + 0^2}d\phi = \frac{1}{2\pi}\int_0^{2\pi} u(x_0 + a\cos(\phi), y_0 + a\sin(\phi))d\phi\,.$$

□

Said differently, the value of u at $\boldsymbol{x}_0$ is the average of the values of u on the boundary of the disc $B_a(\boldsymbol{x}_0)$. It is infact a simple extension to show that the value of u at $\boldsymbol{x}_0$ is also the average of the value of u in the disc $B_a(\boldsymbol{x}_0)$, i.e.

$$u(\boldsymbol{x}_0) = \frac{1}{\pi a^2}\iint_{B_a(\boldsymbol{x}_0)} u(\boldsymbol{x}')dV = \frac{1}{\pi a^2}\int_0^a\int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))r\,drd\theta\,.$$

Since u satisfies the mean value property discussed above,

$$\frac{1}{\pi a^2}\int_0^a\int_0^{2\pi} u(x_0 + r\cos(\theta), y_0 + r\sin(\theta))\,d\theta\,r\,dr = \frac{1}{\pi a^2}\int_0^a u(\boldsymbol{x}_0)2\pi r\,dr = \frac{2}{a^2}u(\boldsymbol{x}_0)\int_0^a r\,dr = u(\boldsymbol{x}_0)\,.$$

There are a couple of immediate consequences of the mean value property. If u satisfies the mean value property then u satisfies the strong maximum principle which we prove below. Moreover, if u satisfies the mean value property, then u is smooth which is left as an exercise in the practice problem set.

Theorem 11.3. *Suppose u is continuous, satisfies the mean value property in D, and achieves it's maximum value at $\boldsymbol{x}_0 \in D$, then u is a constant.*

Remark 11.1. *Note, that this automatically implies the strong maximum principle for harmonic functions since every harmonic function satisfies the mean value property.*

To prove this theorem, we need the following lemma from topology for which the proof is left as an exercise.

Lemma 11.1. *Clopen sets. Suppose X is a connected set and if A is both open and closed relative to X then either $A = \emptyset$ or $A = X$.*

Proof. Suppose the maximum value of u achieved at $\boldsymbol{x}_0 \in D$ is M, i.e. $u(\boldsymbol{x}_0) = M$. Let $A = \{\boldsymbol{x} : u(\boldsymbol{x}) = M\}$ is the subset of D on which u achieves it's maximum. Since $\{M\}$ is a closed set in R and $A = u^{-1}(\{M\})$ is the inverse image of closed set under a continuous function, we conclude that A is closed. Suppose $\boldsymbol{x} \in A$, then $u(\boldsymbol{x}) = M$. Then by the mean value property

$$u(\boldsymbol{x}) = \frac{1}{\pi a^2} \iint_{B_a(\boldsymbol{x})} u(\boldsymbol{x}') dV .$$

The only way the above equality can hold is if $u(\boldsymbol{x}') = M$ for all $\boldsymbol{x}' \in B_a(\boldsymbol{x})$. We prove it by contradiction. Suppose that is not the case, then there exists $\boldsymbol{x}_1 \in B_a(\boldsymbol{x})$ such that $u(\boldsymbol{x}_1) = M - \epsilon$ for some $\epsilon > 0$. By the continuity of u there exists a small enough disc centered at $\boldsymbol{x}_1$, $B_\delta(\boldsymbol{x}_1)$ such that

$$\boldsymbol{x}' \in B_\delta \boldsymbol{x}_1 \implies u(\boldsymbol{x}') \leq M - \epsilon/2$$

Then,

$$M = u(\boldsymbol{x}) = \frac{1}{\pi a^2} \iint_{B_a(\boldsymbol{x})} u(\boldsymbol{x}') dV \tag{84}$$

$$= \frac{1}{\pi a^2} \left(\iint_{B_\delta(\boldsymbol{x}_1)} u(\boldsymbol{x}') dV + \iint_{B_a(\boldsymbol{x}) \backslash B_\delta(\boldsymbol{x}_1)} u(\boldsymbol{x}') dV \right) \tag{85}$$

$$\leq \frac{1}{\pi a^2} \left((M - \epsilon/2) \cdot \pi \delta^2 + M \cdot \left(\pi a^2 - \pi \delta^2 \right) \right) < M \tag{86}$$

Thus if $u(\boldsymbol{x}) = M$, there exists a such that $\boldsymbol{x}' \in B_a(\boldsymbol{x}) \implies u(\boldsymbol{x}') = M$, i.e. if $\boldsymbol{x} \in A$, then $B_a(\boldsymbol{x}) \in A$. This shows that the set A is open. Since the set D is connected, and A is both open and closed relative to D, using Lemma 11.1, we conclude that $A = D$, since $A \neq \emptyset$ as $\boldsymbol{x}_0 \in A$. ◻

11.6.2 Smoothness of harmonic functions

Another immediate consequence of Poisson's formula is that harmonic functions are smooth.

Theorem 11.4. *Suppose u is harmonic in D, then u is smooth.*

Proof. To show that above result, we will show that $\partial_x u(\boldsymbol{x})$ exists. The proof for ∂_y and higher order derivatives will proceed in a similar manner. Let $B_a(\boldsymbol{x}_0) \in D$ and suppose $\boldsymbol{x} \in B_{a/2}(\boldsymbol{x}_0)$. Then by Poisson's formula, the solution $u(\boldsymbol{x})$ in the disc $B_a(\boldsymbol{x}_0)$ is given by

$$u(\boldsymbol{x}) = \frac{a^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2}{2\pi} \int_0^{2\pi} \frac{u(x_0 + a\cos(\theta), y_0 + a\sin(\theta))}{|\boldsymbol{x} - \boldsymbol{x}'|^2} d\theta$$

where $\boldsymbol{x}' - \boldsymbol{x}_0 = (a\cos\theta, a\sin\theta)$. Since $\boldsymbol{x} \in B_{a/2}(\boldsymbol{x}_0)$,

$$|\boldsymbol{x} - \boldsymbol{x}'| = |\boldsymbol{x}' - \boldsymbol{x}_0 - (\boldsymbol{x} - \boldsymbol{x}_0)| \geq |\boldsymbol{x}' - \boldsymbol{x}_0| - |\boldsymbol{x} - \boldsymbol{x}_0| \geq a - \frac{a}{2} = \frac{a}{2} ,$$

Moreover

$$\partial_x \left(\frac{a^2 - |\boldsymbol{x} - \boldsymbol{x}_0|^2}{|\boldsymbol{x} - \boldsymbol{x}'|^2} \right) = \frac{|\boldsymbol{x} - \boldsymbol{x}'|^2 \cdot (-2(x - x_0)) - (a^2 - |\boldsymbol{x} - \boldsymbol{x}'^2|) \cdot (x - x')}{|\boldsymbol{x} - \boldsymbol{x}'|^4} \leq \frac{M}{a^4} \quad \forall \boldsymbol{x}' \in \partial B_a(\boldsymbol{x}_0) .$$

Thus,

$$\partial_x u(\boldsymbol{x}) = \frac{1}{2\pi} \partial_x \int_0^{2\pi} u(x_0 + a\cos(\theta), y_0 + a\sin(\theta)) \frac{a - |\boldsymbol{x} - \boldsymbol{x}_0|^2}{|\boldsymbol{x} - \boldsymbol{x}'|^2} d\theta \tag{87}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + a\cos(\theta), y_0 + a\sin(\theta)) \partial_x \left(\frac{a - |\boldsymbol{x} - \boldsymbol{x}_0|^2}{|\boldsymbol{x} - \boldsymbol{x}'|^2} \right) d\theta \tag{88}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + a\cos(\theta), y_0 + a\sin(\theta)) \frac{|\boldsymbol{x} - \boldsymbol{x}'|^2 \cdot (-2(x - x_0)) - (a^2 - |\boldsymbol{x} - \boldsymbol{x}'^2|) \cdot (x - x')}{|\boldsymbol{x} - \boldsymbol{x}'|^4} d\theta \tag{89}$$

We could switch the order of differentiation and integration as the integrals

$$\frac{1}{2\pi} \partial_x \int_0^{2\pi} u(x_0 + a\cos(\theta), y_0 + a\sin(\theta)) \frac{a - |\boldsymbol{x} - \boldsymbol{x}_0|^2}{|\boldsymbol{x} - \boldsymbol{x}'|^2} d\theta$$

and

$$\int_0^{2\pi} u(x_0 + a\cos(\theta), y_0 + a\sin(\theta)) \frac{|\boldsymbol{x}-\boldsymbol{x}'|^2 \cdot (-2(x-x_0)) - (a^2 - |\boldsymbol{x}-\boldsymbol{x}'^2|) \cdot (x - x')}{|\boldsymbol{x}-\boldsymbol{x}'|^4} d\theta$$

converge absolutely as the integrands are bounded in both cases ($|\boldsymbol{x}-\boldsymbol{x}'| > a/2$). The kernel in Poisson's formula

$$\frac{a^2 - |\boldsymbol{x}-\boldsymbol{x}_0|^2}{|\boldsymbol{x}-\boldsymbol{x}'|^2},$$

is a smooth function for all $\boldsymbol{x} \in B_{a/2}(\boldsymbol{x}_0)$ when $\boldsymbol{x}' \in \partial B_a(\boldsymbol{x}_0)$ where one can obtain trivial bounds uniform in $\boldsymbol{x}'$ for derivatives of all orders which allows us to commute the order of differentiation and integration thereby allowing us to conclude that u is smooth. □

In both, the proof of the mean value theorem and hence consequently the strong maximum principle and the smoothness of harmonic function, we have strongly used the fact that the unique solution to Laplace's equation with given boundary conditions is given by Poisson's formula. Next we prove the result.

Theorem 11.5. *Suppose*

$$u = \int_0^{2\pi} P(r, \theta - \phi) h(\phi) \frac{d\phi}{2\pi},$$

where

$$P(r,\theta) = \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta) + r^2}.$$

Then

$$\Delta u = 0, \quad \text{and} \quad \lim_{(r,\theta)\to(a,\theta_0)} u(r,\theta) = h(\theta_0)$$

Proof. We note alternate forms of the kernel $P(r,\theta)$,

$$P(r,\theta) = 1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n\theta) = \frac{a^2 - r^2}{(a-r)^2 + 4ar\sin^2\left(\frac{\theta}{2}\right)}$$

Each of the functions $\left(\frac{r}{a}\right)^n \cos(n\theta)$ are harmonic. Infact, they are the separation of variables solution in polar coordinates for Laplace's equation. Furthermore, the series

$$P(r,\theta) = 1 + 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n\theta),$$

converges absolutely $r < a$ since it is bounded above by the geometric series $\left(\frac{r}{a}\right)^n$. Then

$$\partial_r P(r,\theta) = 2\partial_r \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n\theta).$$

In order to switch the order of taking the derivative and computing the sum, a sufficient condition is to ensure that

$$2\sum_{n=1}^{\infty} \partial_r \left(\frac{r}{a}\right)^n \cos(n\theta) = 2\sum_{n=1}^{\infty} \frac{n}{a}\left(\frac{r}{a}\right)^{n-1} \cos(n\theta),$$

which is clearly the case since $r < a$. A similar argument would allow us to commute ∂_θ

$$\partial_\theta 2\sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n\theta) = 2\sum_{n=1}^{\infty} \partial_\theta \left(\frac{r}{a}\right)^n \cos(n\theta),$$

and higher order derivatives as well. Thus,

$$\Delta P(r,\theta) = 2\Delta \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n \cos(n\theta) = 2\sum_{n=1}^{\infty} \Delta \left(\frac{r}{a}\right)^n \cos(n\theta) = 0.$$

We are almost ready to prove that u is harmonic.

$$\Delta u = \Delta \int_0^{2\pi} P(r,\theta-\phi) h(\phi) \frac{d\phi}{2\pi}.$$

Using techniques discussed in the proof of Theorem 11.4, we can switch the order of differentiation integration in the equation above, we get

$$\Delta_{r,\theta} u = \Delta_{r,\theta} \int_0^{2\pi} P(r, \theta - \phi) h(\phi) \frac{d\phi}{2\pi} = \int_0^{2\pi} \Delta_{r,\theta} P(r, \theta - \phi) h(\phi) \frac{d\phi}{2\pi} = 0\,,$$

where $\Delta_{r,\theta}$ denotes the Laplacian with respect to the coordinates (r, θ). Now all we need to show is that u achieves the right boundary data. If we carefully look at the function $P(r, \theta)$, we see that

$$P(a, \theta) = 0 \quad \forall \theta \neq 0\,,$$

and $P(a, 0) = \infty$. This looks awfully close to heat kernel. In fact in the limit $r \to a$, $P(r, \theta)$ behaves like a dirac at $\theta = 0$. We first reformulate the limit we are trying to compute and make two more observations about the kernel $P(r, \theta)$. Firstly,

$$\int_0^{2\pi} P(r, \theta - \phi) \frac{d\phi}{2\pi} = 0 \quad \forall r < a$$

This follows from the series expansion of $P(r, \theta)$ given by

$$P(r, \theta) = 1 + 2 \sum_{n=1}^{\theta} \left(\frac{r}{a}\right)^n \cos(n\theta)\,,$$

and the fact that

$$\int_0^{2\pi} \cos(n(\theta - \phi)) \frac{d\phi}{2\pi} = 0\,.$$

And secondly

$$P(r, \theta) = \frac{a^2 - r^2}{(a - r)^2 + 4ar \sin^2\left(\frac{\theta}{2}\right)} \geq 0\,.$$

Then

$$u(r, \theta) - h(\theta_0) = \int_0^{2\pi} P(r, \theta - \phi) h(\phi) \frac{d\phi}{2\pi} = \int_0^{2\pi} P(r, \theta - \phi) \left(h(\phi) - h(\theta_0)\right) \frac{d\phi}{2\pi}\,,$$

since $\int_0^{2\pi} P(r, \theta - \phi) d\phi / 2\pi = 1$. Thus, the statement

$$\lim_{r \to a, \theta \to \theta_0} u(r, \theta) = h(\theta_0)\,,$$

is equivalent to showing

$$\lim_{r \to a, \theta \to \theta_0} |u(r, \theta) - h(\theta_0)| = \lim_{r \to a, \theta \to \theta_0} \left| \int_0^{2\pi} P(r, \theta - \phi) \left(h(\phi) - h(\theta_0)\right) \frac{d\phi}{2\pi} \right| = 0\,.$$

How do we proceed to prove this? If we look at the behaviour of P, when ϕ is separated from θ_0, then

$$\lim_{r \to a, \theta \to \theta_0} P(r, \theta - \phi) = 0\,.$$

When ϕ is close to θ_0, P blows up in the limit but

$$\lim_{r \to a, \theta \to \theta_0} \left(h(\phi) - h(\theta_0)\right) = 0\,.$$

So we split the integration domain into two parts. Suppose $\varepsilon > 0$. Using the continuity of h, let δ be such that

$$|\phi - \theta_0| < \delta \implies |h(\phi) - h(\theta_0)| < \epsilon\,.$$

Then,

$$\begin{aligned}
\left| \int_0^{2\pi} P(r, \theta - \phi) \left(h(\phi) - h(\theta_0)\right) \frac{d\phi}{2\pi} \right| &= \left| \int_{|\phi - \theta_0| < \delta} P(r, \theta - \phi) \left(h(\phi) - h(\theta_0)\right) \frac{d\phi}{2\pi} + \int_{|\phi - \theta_0| \geq \delta} P(r, \theta - \phi) \left(h(\phi) - h(\theta_0)\right) \frac{d\phi}{2\pi} \right| \\
&\leq \left| \int_{|\phi - \theta_0| < \delta} P(r, \theta - \phi) \left(h(\phi) - h(\theta_0)\right) \frac{d\phi}{2\pi} \right| + \left| \int_{|\phi - \theta_0| \geq \delta} P(r, \theta - \phi) \left(h(\phi) - h(\theta_0)\right) \frac{d\phi}{2\pi} \right| \quad (\triangle\text{ineq}) \\
&\leq \int_{|\phi - \theta_0| < \delta} \left| P(r, \theta - \phi) \left(h(\phi) - h(\theta_0)\right) \right| \frac{d\phi}{2\pi} + \int_{|\phi - \theta_0| \geq \delta} \left| P(r, \theta - \phi) \left(h(\phi) - h(\theta_0)\right) \right| \frac{d\phi}{2\pi} \quad \left(\left|\int f\right| \leq \int |f|\right) \\
&= I + II
\end{aligned}$$

Let's focus on the first integral.

$$\begin{aligned}
I &= \int_{|\phi-\theta_0|<\delta} |P(r,\theta-\phi)\,(h(\phi)-h(\theta_0))|\,\frac{d\phi}{2\pi} \\
&= \int_{|\phi-\theta_0|<\delta} P(r,\theta-\phi)\,|(h(\phi)-h(\theta_0))|\,\frac{d\phi}{2\pi} \quad (P>0) \\
&\leq \int_{|\phi-\theta_0|<\delta} P(r,\theta-\phi)\varepsilon\frac{d\phi}{2\pi} \quad \text{(continuity of } h\text{)} \\
&\leq \varepsilon \int_0^{2\pi} P(r,\theta-\phi)\frac{d\phi}{2\pi} \\
&= \varepsilon \quad (\int_0^{2\pi} P d\phi = 1)\,.
\end{aligned}$$

And now the second integral. First we note that since h is continuous $|h| \leq M$ for some constant M. Secondly, we change $\phi \to \phi - \theta_0$, to rewrite the second integral as

$$\begin{aligned}
II &= \int_\delta^{2\pi-\delta} |P(r,\theta-\theta_0-\phi)\,(h(\phi+\theta_0)-h(\theta_0))|\,\frac{d\phi}{2\pi} \\
&\leq \frac{2M}{2\pi}\int_\delta^{2\pi-\delta} P(r,\theta-\theta_0-\phi)d\phi \quad (|h|<M \quad \text{and} \quad P>0)
\end{aligned}$$

Now our goal is to choose r, θ sufficiently close to a, θ_0 such that the above integral can be bounded by ε (upto a constant). For that, let θ satisfy $|\theta-\theta_0| < \delta/2$. Then $|\theta-\theta_0-\phi| \in (\delta/2, 2\pi-\delta/2)$ whenever $\phi \in (\delta, 2\pi-\delta)$ Moreover

$$P(r,\theta-\theta_0-\phi) = \frac{a^2-r^2}{(a-r)^2 + 4ar\sin^2\left(\frac{\theta-\theta_0-\phi}{2}\right)} \leq \frac{a^2-r^2}{4ar\sin^2\left(\frac{\delta}{4}\right)}$$

From the above expression, it is clear that $\exists r_0$ such that $r_0 < r < a$ so that

$$P(r,\theta-\theta_0-\phi) \leq \varepsilon \quad \forall \phi \in (\delta, 2\pi-\delta) \quad \text{and} \quad |\theta-\theta_0| < \frac{\delta}{2}\,.$$

Plugging all of this back into the estimate of the second integral, we get

$$II \leq \frac{2M}{2\pi}\int_\delta^{2\pi-\delta} P(r,\theta-\theta_0-\phi)d\phi \leq 2M\varepsilon\,.$$

Combining the two estimates, the result follows. □

12 Green's identities and Laplace's equations in 3D

The equivalent of integration by parts in higher dimensions are often referred to as Green's identities and follow from the divergence theorem. Let u, v be two functions defined on $D \subset \mathbb{R}^3$, then the divergence theorem applied to $u\nabla v$ gives us

$$\begin{aligned}
\iint_{\partial D} u\nabla v \cdot \boldsymbol{n} dS &= \iiint_D \nabla \cdot (u\nabla v) \\
\iint_{\partial D} u\frac{\partial v}{\partial n} dS &= \iiint_D (\nabla u \cdot \nabla v + u\nabla \cdot \nabla v)\, dV \\
&= \iiint_D \nabla u \cdot \nabla v dV + \iiint_D u\Delta v dV\,.
\end{aligned}$$

The above identity is often referred to as *Green's first identity.* Replacing the role of u and v in the above equation and subtracting it from the equation above, we get *Green's second identity*, given by

$$\iint_{\partial D}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS = \iiint_D (u\Delta v - v\Delta u)\, dV$$

12.1 Uniqueness using energy methods

Applying Green's first identity to the functions u and u, we get

$$\iint_{\partial D} u \frac{\partial u}{\partial n} dS = \iiint_D |\nabla u|^2 \, dV + \iiint_D u \Delta u dV \,.$$

This gives us an energy method based proof of uniqueness of solutions to the Dirichlet and Neumann boundary value problems in three dimensions. Suppose u is harmonic in D and $u = 0$ on ∂D, then $u \equiv 0$ in the interior D. Since u is harmonic $\Delta u = 0$ and the last term on the right in the above equation is 0 and since u is 0 on the boundary ∂D, the term on the left is also 0. Thus, we get

$$\iiint_D |\nabla u|^2 dV = 0 \implies |\nabla u| \equiv 0 \text{ in } D \,,$$

and therefore u is a constant in D. However, since $u = 0$ on the boundary, we conclude that $u \equiv 0$ in D.

12.2 Mean value property in higher dimensions

In two dimensions, we proved mean value theorem using Poisson's formula. One can proceed to prove mean value theorem's in a similar fashion by constructing a Green's function for a "disk" in n dimensions as well. However, let us look at an alternate proof for the mean value property. Suppose we apply divergence theorem for the vector field ∇u where D is the ball $B_a(0)$. Then

$$\iint_{\partial B_a(0)} \frac{\partial u}{\partial n} dS = \iiint_{B_a(0)} \Delta u dV \,.$$

If u is harmonic in $B_a(0)$, we get

$$\iint_{\partial B_a(0)} \frac{\partial u}{\partial n} dS = 0 \,.$$

Furthermore on the boundary of the sphere $\partial B_a(0)$ given by $x^2+y^2+z^2 = a^2$, the normal at (x, y, z) is given by $\boldsymbol{n}(\boldsymbol{x}) = \frac{\boldsymbol{x}}{|\boldsymbol{x}|} = \hat{r}$. Thus

$$\iint_{\partial B_a(0)} \frac{\partial u}{\partial n} dS = \iint_{\partial B_a(0)} \frac{\partial u}{\partial r} dS = 0 \,.$$

Suppose $(x, y, z) = a(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ be a parmaterization of the surface $\partial B_a(0)$, then

$$\iint_{\partial B_a(0)} \frac{\partial u}{\partial r} , dS = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} u_r(a, \theta, \phi) a^2 \sin(\theta) \, d\theta d\phi = 0$$

Dividing throughout by $4\pi a^2$ and exchanging the order of differentiation and integration, we get

$$\frac{1}{4\pi} \partial_r \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} u(a, \theta, \phi) \sin(\theta) \, d\theta d\phi = 0 \,.$$

Just a tautological remark, the above expression is $f'(a)$ where $f(r)$ is the function defined by

$$f(r) = \frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} u(r, \theta, \phi) \sin(\theta) \, d\theta d\phi \,.$$

Thus, $f'(a) = 0$ for all a such that u is harmonic in $B_a(0)$. In particular, $f'(r) = 0$ for all $r < a$ if u is harmonic in $B_a(0)$. Thus, f is a constant and

$$lim_{r \to 0} f(r) = f(a) = \frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} u(a, \theta, \phi) \sin(\theta) \, d\theta d\phi \,.$$

$$\frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} u(a, \theta, \phi) \sin(\theta) \, d\theta d\phi = \lim_{r \to 0} \frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} u(r, \theta, \phi) \sin(\theta) \, d\theta d\phi \qquad = u(0)$$

The last equality follows from the continuity of u. Suppose r_0 is small enough so that $r < r_0 \implies$

$$|u(r, \theta, \phi) - u(0)| < \epsilon \,,$$

for all $r < r_0$, $\theta \in [0, \pi]$ and $\phi \in (0, 2\pi]$. Then

$$\begin{aligned}
\frac{1}{4\pi}\left|\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} u(r,\theta,\phi)\sin(\theta)\,d\theta d\phi - u(0)\right| &= \left|\frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(u(r,\theta,\phi)-u(0))\sin(\theta)\,d\theta d\phi\right| \quad \text{Since } \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\sin(\theta)d\theta d\phi = 4\pi \\
&\leq \frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}|u(r,\theta,\phi)-u(0)|\sin(\theta)\,d\theta d\phi \quad \text{Since } \left|\int f\right| \leq \int |f| \\
&\leq \frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\epsilon\sin(\theta)d\theta d\phi \quad \forall r < r_0 \\
&= \epsilon\,,
\end{aligned}$$

which proves the result.

Once we have proved the mean value theorem, we get the strong maximum principle follows from Theorem 11.3 which only uses the clopen argument. Furthermore, uniqueness follows from the maximum principle too.

12.3 Connection between PDEs and calculus of variations

Harmonic functions on a domain with specific boundary conditions can be also be understood as minimizers of the Dirichlet energy where the admissible class of functions is restricted to the functions which achieve the right boundary conditions.

Consider the family of functions

$$\mathcal{F} : \{w(\boldsymbol{x}) : \text{such that } w(\boldsymbol{x}) = h(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \partial D\}$$

Then $u(\boldsymbol{x})$ is the minimizer of the Dirichlet energy

$$E[w] = \iint\limits_{D} |\nabla w|^2 dV\,,$$

constrained to the family of functions $\mathcal{F}$ if and only if $u(\boldsymbol{x})$ is harmonic and satisfies the boundary conditions $u(\boldsymbol{x}) = h(\boldsymbol{x})$ on ∂D. For each function w, $E[w]$ is a number. Suppose we compute $E[w]$ for all functions $w \in \mathcal{F}$, then the function for which the minimum value is achieved for $E[w]$ is precisely the unique harmonic function in the family $\mathcal{F}$. Further the converse is also true – the energy of all functions w is more than the energy of the harmonic function $w \in \mathcal{F}$. Here is the precise statement.

Theorem 12.1. *u satisfies*

$$\Delta u = 0 \quad \boldsymbol{x} \in D\,, \quad u(\boldsymbol{x}) = h(\boldsymbol{x})\,\boldsymbol{x} \in \partial D,$$

if and only if $u(\boldsymbol{x})$ satisfies

$$u(\boldsymbol{x}) = argmin_{w\in\mathcal{F}} E[w]\,,$$

i.e. u is the function in $\mathcal{F}$ which minimizes the Dirichlet energy

Proof. Suppose u satisfies the PDE,

$$\Delta u = 0 \quad \boldsymbol{x} \in D\,, \quad u(\boldsymbol{x}) = h(\boldsymbol{x}) \quad \boldsymbol{x} \in \partial D\,.$$

Clearly $u \in \mathcal{F}$ since u is h on the boundary ∂D. Suppose $v \in \mathcal{F}$. Let $w = u - v$. Then $w(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \partial D$ since both u and v satisfy $u = v = h$ for $\boldsymbol{x} \in \partial D$. Then applying Green's first identity to u and w, we get

$$\iint_D w \cdot \Delta u\, dV = \iint_{\partial D} w\frac{\partial u}{\partial n} dS - \iint_D \nabla w \cdot \nabla u dS\,.$$

$\Delta u = 0$ in D, $w = 0$ on ∂D, thus the above equation gives us

$$\iint_D \nabla w \cdot \nabla u dV = 0\,,$$

Then

$$E[v] = E[u-w] = \frac{1}{2}\iint_D |\nabla(u-w)|^2 dV = \frac{1}{2}\iint_D |\nabla u|^2 dV + \frac{1}{2}\iint_D |\nabla v|^2 dV - \iint_D \nabla w \cdot \nabla u dV = E[u] + E[w]\,.$$

Thus, $E[v] \geq E[u]$ since $E[w] > 0$.

We now show the other direction. Suppose u minimizes the Dirichlet energy in F. Then for any function which is 0 on the boundary ∂D, $u + \epsilon w \in \mathcal{F}$ for all ϵ. Furthermore, since u minimizes the dirichlet energy, the function

$$f(\epsilon) = E[u + \epsilon w] \,,$$

achieves it's minimum value at $\epsilon = 0$ for all functions w which are 0 on the boundary, i.e. $f'(0) = 0$.

$$\begin{aligned} f(\epsilon) &= \frac{1}{2} \iint_D |\nabla(u + \epsilon w)|^2 \, dV \\ &= \frac{1}{2} \iint_D |\nabla u|^2 \, dV + \epsilon \iint_D \nabla w \cdot \nabla u \, dV + \epsilon^2 \frac{1}{2} |\nabla w|^2 \, dV \\ &= E[u] + \epsilon \iint_D \nabla u \cdot \nabla w \, dV + \epsilon^2 E[w] \,. \end{aligned}$$

Then

$$f'(\epsilon) = \iint_D \nabla u \cdot \nabla w \, dV + 2\epsilon E[w] \,.$$

Finally, $f'(0) = 0$ implies

$$f'(0) = \iint_D \nabla u \cdot \nabla w \, dV = 0 \,,$$

for all functions w which are 0 on the boundary. Using green's first identity,

$$\iint_D \nabla u \cdot \nabla w \, dV = \iint_{\partial D} w \frac{\partial u}{\partial n} \, dS - \iint_D \Delta u \cdot w \, dV = 0 \,.$$

Since $w = 0$ on the boundary ∂D, we conclude

$$\iint_D \Delta u \cdot w \, dV = 0$$

for all functions $w = 0$ on the boundary. From this, we conclude that $\Delta u \equiv 0$ for $\boldsymbol{x} \in D$. Suppose not. Suppose there exists $\boldsymbol{x}_0 \in D$ such that $\Delta u(\boldsymbol{x}_0) = \delta > 0$ (the proof of negative δ follows in a similar fashion). By the continuity of the Laplacian, we conclude that $\Delta u(\boldsymbol{x}) > 0$ for all $\boldsymbol{x} \in B_{r_0}(\boldsymbol{x}_0)$ where r_0 is sufficiently small and $B_{r_0}(\boldsymbol{x}_0) \subset D$. Then, we choose w such that $w = 1$ in $B_{r_0/2}(\boldsymbol{x}_0)$, $w > 0$ for $\boldsymbol{x} \in B_{r_0}(\boldsymbol{x}_0)$ and $w = 0$ for $\boldsymbol{x}$ in $D \setminus B_{r_0}(\boldsymbol{x}_0)$. We can construct such functions using bump functions discussed in the practice problem set. Then for this particular function w,

$$\iint_D w \Delta u \, dV = \iint_{B_{r_0}(\boldsymbol{x}_0)} w \Delta u \, dV > 0 \,,$$

which is a contradiction. $\square$

12.4 Representation theorem

An important consequence of Green's identities is that any harmonic function u in the volume is completely described by two functions defined on the boundary u and $\frac{\partial u}{\partial n}$. If we have both the functions u and $\frac{\partial u}{\partial n}$ avaiable on the boundary, we can use the representation formula below to compute u at any point in the interior.

Theorem 12.2. *If $\Delta u = 0$, then*

$$u(\boldsymbol{x}_0) = \iint_{\partial D} -u(\boldsymbol{x}) \frac{\partial}{\partial n} \left(\frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{x}_0|} \right) + \frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{x}_0|} \frac{\partial u}{\partial n} dS \,.$$

Proof. We start off with Green's second identity applied to two functions u, v.

$$\iint_D (u \Delta v - v \Delta u) \, dV = \iint_{\partial D} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \, dS \,.$$

Since u is harmonic the integral of $v \Delta u$ is 0. The statement of te above theorem want's us to choose

$$v(\boldsymbol{x}) = \frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{x}_0|} \,,$$

which is infact a radially symmetric solution to Laplace's equation in three dimensions for all $\boldsymbol{x}$ except $\boldsymbol{x} = \boldsymbol{x}_0$. Thus, instead of applying Green's second identity to the functions u and v on the domain D, we apply it on the domain $D_\epsilon = D \setminus B_\epsilon(\boldsymbol{x}_0)$.

In D_ϵ, v also satisfies $\Delta v = 0$. Furthermore ∂D_ϵ consists of two parts $\partial D_\epsilon = \partial D \cup \partial B_\epsilon(\boldsymbol{x}_0)$, where the normal to $\partial B_\epsilon(\boldsymbol{x}_0)$ is facing inward toward $\boldsymbol{x}_0$ and the normal to ∂D is the regular outward facing normal. Then

$$0 = \iint_{D_\epsilon} (u\Delta v - v\Delta u) dV = \iint_{\partial D_\epsilon} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS \tag{90}$$

$$= \iint_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS + \iint_{\partial B_\epsilon(\boldsymbol{x}_0)} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS\,. \tag{91}$$

□

On $\partial B_\epsilon(\boldsymbol{x}_0)$, if $\boldsymbol{x} - \boldsymbol{x}_0 = \varepsilon\hat{r}$ then $\boldsymbol{n} = -hatr$, where $\hat{r}$ is the unit radial vector with origin at $\boldsymbol{x}_0$. Furthermore $dS = \varepsilon^2 \sin(\theta) d\theta d\phi$, and,

$$v = \frac{1}{4\pi\varepsilon}\,, \quad \frac{\partial v}{\partial n} = -\frac{1}{4\pi}\frac{(\boldsymbol{x} - \boldsymbol{x}_0)\cdot \boldsymbol{n}}{|\boldsymbol{x} - \boldsymbol{x}_0|^3} = \frac{1}{4\pi\varepsilon^2}\,.$$

Then,

$$\iint_{\partial B_\varepsilon(\boldsymbol{x}_0)} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \left(u\frac{1}{4\pi\varepsilon^2} - \frac{1}{4\pi\varepsilon}\frac{\partial u}{\partial n} \right) \varepsilon^2 \sin(\theta) d\theta d\phi$$

$$\lim_{\varepsilon\to 0}\iint_{\partial B_\varepsilon(\boldsymbol{x}_0)} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS = \lim_{\varepsilon\to 0}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \left(u\frac{1}{4\pi} - \varepsilon\frac{1}{4\pi}\frac{\partial u}{\partial n} \right) \sin(\theta) d\theta d\phi$$

$$= u(\boldsymbol{x}_0)$$

Taking the limit as $\varepsilon \to 0$ in equation (91), we get

$$0 = \iint_{\partial D} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} \right) dS + u(\boldsymbol{x}_0)$$

$$u(\boldsymbol{x}_0) = \iint_{\partial D} \left(-u(\boldsymbol{x})\frac{\partial}{\partial n}\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|} + \frac{\partial u}{\partial n}(\boldsymbol{x})\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|} \right) dS\,.$$

The above representation theorem is really powerful. It says that all the information one needs about harmonic functions lives on the boundary of the domain. Given u on the boundary, if one could compute $\frac{\partial u}{\partial n}$ on the boundary as well, then the above equation gives a formula for u everywhere in the interior. Furthermore, one could also use the representation theorem to conclude that the solution u is smooth in the interior with appealing to Poisson's formula.

12.5 Green's functions

The above formula uses the free space Green's function for representing u in the interior as a function of it's values on the boundary. At this point, it is natural to ask if we can directly compute the solution to the Dirichlet problem by appropriately modifying v in order to eliminate the $v\frac{\partial u}{\partial n}$ term in the representation. This can indeed be done, and the resulting v which satisfies the above property is referred to as the domain Green's function. We have already done this computation for a disk although using separation of variables.

Definition 12.1. *Domain Green's function. Given a simply connected bounded domain D, the domain Green's function for the operator $-\Delta$ on D is a function which satisfies the following properties for every $\boldsymbol{x}_0 \in D$.*

- $$G(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|} + H(\boldsymbol{x}, \boldsymbol{x}_0)\,,$$

  where $H(\boldsymbol{x}, \boldsymbol{x}_0$ as a function of $\boldsymbol{x}$ satisfies Laplace's equation in D for each $\boldsymbol{x}_0$.

- $G(\boldsymbol{x}, \boldsymbol{x}_0) = 0\,, \quad \forall \boldsymbol{x} \in \partial D$

If we could indeed find such a function, then we have a complete solution for the Dirichlet problem for Laplace's equation in D.

Theorem 12.3. *If $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is the domain Green's function for D, and if u satisfies Laplace's equation in D, then*

$$u(\boldsymbol{x}_0) = \iint_{\partial D} u(\boldsymbol{x})\frac{\partial G(\boldsymbol{x}, \boldsymbol{x}_0)}{\partial n} dS\,.$$

The proof of the above statement follows exactly the same way as the derivation of the representation formula. Clearly our Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0) = v(\boldsymbol{x}) + H(\boldsymbol{x}, \boldsymbol{x}_0)$. Plugging it back into the representation formula for u, we get

$$\begin{aligned}
u(\boldsymbol{x}_0) &= \iint_{\partial D} \left(u \left(\frac{\partial G}{\partial n} - \frac{\partial H}{\partial n} \right) - \frac{\partial u}{\partial n} (G - H) \right) dS \,, \\
&= \iint_{\partial D} \left(u \frac{\partial G}{\partial n} - \frac{\partial u}{\partial n} G \right) dS - \iint_{\partial D} \left(u \frac{\partial H}{\partial n} - \frac{\partial u}{\partial n} H \right) dS \\
&= \iint_{\partial D} u \frac{\partial G}{\partial n} \, dS - \iint_{\partial D} \left(u \frac{\partial H}{\partial n} - \frac{\partial u}{\partial n} H \right) dS \quad (G = 0 \text{ on } \partial D) \\
&= \iint_{\partial D} u \frac{\partial G}{\partial n} \, dS - \iint_{D} (u \Delta H - H \Delta u) \, dV \quad (\text{Green's second identity}) \\
&= \iint_{\partial D} u \frac{\partial G}{\partial n} \, dS \quad (\text{Since } u \text{ and } H \text{ are harmonic in } D)
\end{aligned}$$

Unfortunately, finding the Green's function for an arbitrary domain turns out to be a more difficult task than solving the Dirichlet problem on the domain and other than very simple domains, the Green's function cannot be explicitly computed. We have already computed the Green's function for a disk in two dimensions. In the next section, we will compute the domain Green's function when the domain is a half space, say $\{(x, y, z) : z > 0\}$ in three dimesions using the method of images.

For now let us continue with uncovering one nice property of the Green's function. In free space, the Laplace free space Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0)$ represents the potential at $\boldsymbol{x}$ due to a unit "charge" placed at $\boldsymbol{x}_0$. The principle of reciprocity for electrostatics, states that the potential at $\boldsymbol{x}$ due to a point charge at $\boldsymbol{x}_0$ is exactly the same as the potential at $\boldsymbol{x}_0$ due to a point charge at $\boldsymbol{x}$. This is clearly true for the free space Green's function as

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{x}_0|} \,.$$

On the other hand, the domain Green's function $G(\boldsymbol{x}, \boldsymbol{x}_0)$ represents the potential at $\boldsymbol{x}$ inside the domain, due to a charge placed at $\boldsymbol{x}_0$ such that the boundary of the domain D, ∂D is grounded, i.e. at 0 potential. Even in this world, when we ground the boundary, it is natural to expect the principle of reciprocity to hold, and indeed it does hold.

Theorem 12.4. *If $G(\boldsymbol{x}, \boldsymbol{x}_0)$ is the domain Green's function for D, then*

$$G(\boldsymbol{a}, \boldsymbol{b}) = G(\boldsymbol{b}, \boldsymbol{a}) \,.$$

Proof. As all proofs have proceeded in this section, for this proof as well we need to use Green's identities for the right functions u and v and "domain" D. Let $u(\boldsymbol{x}) = G(\boldsymbol{x}, \boldsymbol{a})$ and $v(\boldsymbol{x}) = G(\boldsymbol{x}, \boldsymbol{b})$. If we can show $u(\boldsymbol{b}) = v(\boldsymbol{a})$, then we'd be done. Unfortunately u is not harmonic at $\boldsymbol{a}$ since

$$u(\boldsymbol{x}) = -\frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{a}|} + H_{\boldsymbol{a}}(\boldsymbol{x}) \,,$$

and similarly v is not harmonic at $\boldsymbol{b}$ since

$$v(\boldsymbol{x}) = -\frac{1}{4\pi |\boldsymbol{x} - \boldsymbol{b}|} + H_{\boldsymbol{b}}(\boldsymbol{x}) \,.$$

Here $H_{\boldsymbol{a}}$ and $H_{\boldsymbol{b}}$ are harmonic functions. So we will apply Green's second identity to the functions u and v on the domain $D_\varepsilon = D \setminus (B_\varepsilon(\boldsymbol{a}) \cup B_\varepsilon(\boldsymbol{b}))$

$$\iint_{D_\varepsilon} (u \Delta v - v \Delta u) \, dV = \iint_{\partial D_\varepsilon} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS$$

Since u and v are harmonic in D_ε, the left hand side in the above expression is 0. Secondly, the boundary ∂D_ε is comprised of three parts: $\partial D_\varepsilon = \partial D \cup \partial B_\varepsilon(\boldsymbol{a}) \cup \partial B_\varepsilon(\boldsymbol{b})$, where as always, the normals to $\partial B_\varepsilon(\boldsymbol{a})$ and $\partial B_\varepsilon(\boldsymbol{b})$ are pointing in the inward direction. On the boundary ∂D, again, both u and v are zero since, the Green's function G is zero on the boundary ∂D. Thus,

$$\begin{aligned}
0 &= \iint_{\partial D_\varepsilon} \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS && (92) \\
&= \left(\iint_{\partial D} + \iint_{\partial B_\varepsilon(\boldsymbol{a})} + \iint_{\partial B_\varepsilon(\boldsymbol{b})} \right) \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS && (93) \\
&= \left(\iint_{\partial B_\varepsilon(\boldsymbol{a})} + \iint_{\partial B_\varepsilon(\boldsymbol{b})} \right) \left(u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS && (94)
\end{aligned}$$

We now proceed to compute the integrals on the boundary $\partial B_\varepsilon(\boldsymbol{a})$, the computation for the other boundary proceeds in a similar manner. In the neighborhood of $\partial B_\varepsilon(\boldsymbol{a})$, v is a nice bounded, harmonic function with derivatives of all order. The problematic function is u. Here is an informal analysis to understand what terms will end up mattering. On the boundary $\partial B_\varepsilon(\boldsymbol{a})$, $u \sim O(\varepsilon^{-1}) + \text{smooth}$, and $\frac{\partial u}{\partial n} \sim O(\varepsilon^{-2}) + \text{smooth}$, where smooth in these equations correspond to the contribution of $H_{\boldsymbol{a}}(\boldsymbol{x})$. The surface area element $dS \sim O(\varepsilon^2)$. So the only term that will have a non-trivial contribution in the limit $\varepsilon \to 0$ is

$$\frac{\partial}{\partial n} \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{a}|} \,,$$

and the contributions of all the other terms will vanish in the limit $\varepsilon \to 0$ Let's first focus on $u\frac{\partial v}{\partial n}$ term.

$$\begin{aligned}
\lim_{\varepsilon\to 0}\left|\iint_{\partial B_\varepsilon(a)} u\frac{\partial v}{\partial n} dS\right| &= \lim_{\varepsilon\to 0}\left|\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(-\frac{1}{4\pi\varepsilon}+H_{\boldsymbol{a}}\right)\frac{\partial v}{\partial n}\varepsilon^2\sin(\theta)d\theta d\phi\right| \\
&\le \lim_{\varepsilon\to 0}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left|\left(-\frac{\varepsilon}{4\pi}+\varepsilon^2 H_{\boldsymbol{a}}\right)\frac{\partial v}{\partial n}\sin(\theta)d\theta d\phi\right| \qquad (|\int f| \le \int |f|\ )
\end{aligned}$$

We can change the order taking the limit and integration as the integrand is smooth and bounded owing to the smoothness of $H_{\boldsymbol{a}}$ and $\frac{\partial v}{\partial n}$ on the boundary $\partial B_\varepsilon(\boldsymbol{a})$. Thus

$$\lim_{\varepsilon\to 0}\left|\iint_{\partial B_\varepsilon(a)} u\frac{\partial v}{\partial n} dS\right| \le \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\lim_{\varepsilon\to 0}\left|\left(-\frac{\varepsilon}{4\pi}+\varepsilon^2 H_{\boldsymbol{a}}\right)\frac{\partial v}{\partial n}\sin(\theta)d\theta d\phi\right| = 0$$

We now proceed to $v\frac{\partial u}{\partial n}$. On the boundary $\partial B_\varepsilon(\boldsymbol{a})$, $\boldsymbol{n} = -(\boldsymbol{x}-\boldsymbol{a})/|\boldsymbol{x}-\boldsymbol{a}|$ and

$$\frac{\partial u}{\partial n} = -\frac{(\boldsymbol{x}-\boldsymbol{a})\cdot\boldsymbol{n}}{4\pi|\boldsymbol{x}-\boldsymbol{a}|^3} + \frac{\partial H_{\boldsymbol{a}}}{\partial n} = \frac{1}{4\pi\varepsilon^2} + \frac{\partial H_{\boldsymbol{a}}}{\partial n} \,.$$

Then

$$\begin{aligned}
\lim_{\varepsilon\to 0}\iint_{\partial B_\varepsilon(a)} v\frac{\partial u}{\partial n} dS &= \lim_{\varepsilon\to 0}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\frac{1}{4\pi\varepsilon^2}+\frac{\partial H_{\boldsymbol{a}}}{\partial n}\right)v\varepsilon^2\sin(\theta)d\theta d\phi \\
&= \lim_{\varepsilon\to 0}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(\frac{1}{4\pi}+\varepsilon^2\frac{\partial H_{\boldsymbol{a}}}{\partial n}\right)v\sin(\theta)d\theta d\phi \\
&= v(\boldsymbol{a}) + 0
\end{aligned}$$

The $\varepsilon^2 H_{\boldsymbol{a}}$ term goes to zero as argued before since the we can interchange the order of taking the limit and computing the integral, since the integrand $\frac{\partial H_{\boldsymbol{a}}}{\partial n}\cdot v$ is bounded and smooth. The first term on the other hand converges to $v(\boldsymbol{a})$ owing to the continuity of v at $\boldsymbol{a}$. Combining all of these things, we get

$$\lim_{\varepsilon\to 0}\iint_{\partial B_\varepsilon(\boldsymbol{a})}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS = -v(\boldsymbol{a})$$

Using a similar calculation, we can show that

$$\lim_{\varepsilon\to 0}\iint_{\partial B_\varepsilon(\boldsymbol{b})}\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS = u(\boldsymbol{b})$$

Taking the limit as $\varepsilon \to 0$ in equation (94), we get

$$0 = \lim_{\varepsilon\to 0}\left(\iint_{\partial B_\varepsilon(\boldsymbol{a})} + \iint_{\partial B_\varepsilon(\boldsymbol{b})}\right)\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS = -v(\boldsymbol{a}) + u(\boldsymbol{b}) \,,$$

and we get the desired result. □

12.6 Method of images

As mentioned before, it is not straightforward to compute Green's functions for general domains. However, for fairly simple domains like half-spaces and spheres, we can use the method of images to compute the domain Green's function for them.

As an example, let us consider the half-space $D = \{(x, y, z) : z > 0\}$, whose boundary ∂D is given by $z = 0$. The domain Green's function is the function of the form

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|} + H(\boldsymbol{x}, \boldsymbol{x}_0)\,,$$

where H is harmonic and G satisfies $G((x, y, 0), \boldsymbol{x}_0) = 0$ for all $\boldsymbol{x}_0 = (x_0, y_0, z_0)$ with $z_0 > 0$. A simple way to annihilate the potential due a charge at $\boldsymbol{x}_0$

$$-\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|}\,,$$

on the surface $(x, y, 0)$ is to place an equal and opposite charge at $\boldsymbol{x}_0^* = (x_0, y_0, -z_0)$ whose field is given by

$$H(\boldsymbol{x}, \boldsymbol{x}_0) = \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0^*|}\,.$$

Clearly $\boldsymbol{x}_0^*$ is outside of D, since $\boldsymbol{x}_0 \in D \implies z_0 > 0$ and thus $\boldsymbol{x}_0^*$ in $\mathbb{R}^3 \setminus D$. Thus, $H(\boldsymbol{x}, \boldsymbol{x}_0)$ is a harmonic function in $\boldsymbol{x}$. Furthermore,

$$\begin{aligned} G((x, y, 0), \boldsymbol{x}_0) &= -\frac{1}{4\pi|(x, y, 0) - (x_0, y_0, z_0)|} + \frac{1}{4\pi|(x, y, 0) - (x_0, y_0, -z_0)|} \\ &= -\frac{1}{4\pi\sqrt{((x - x_0)^2 + (y - y_0)^2 + (-z_0)^2)}} + \frac{1}{4\pi\sqrt{((x - x_0)^2 + (y - y_0)^2 + z_0^2)}} = 0 \end{aligned}$$

Thus, the Green's function for the half-space is given by

$$G(\boldsymbol{x}, \boldsymbol{x}_0) = -\frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0|} + \frac{1}{4\pi|\boldsymbol{x} - \boldsymbol{x}_0^*|}\,,$$

where $\boldsymbol{x}_0^* = (x_0, y_0, -z_0)$. The representation theorem for the dirichlet problem for Laplace's equation on the half-space is then given by

$$\begin{aligned} u(\boldsymbol{x}_0) &= \iint_{\partial D} \frac{\partial G(\boldsymbol{x}, \boldsymbol{x}_0)}{\partial n} u(\boldsymbol{x}) dS \\ u(x_0, y_0, z_0) &= \frac{z_0}{2\pi} \int_{x \in \mathbb{R}} \int_{y \in \mathbb{R}} \frac{u(x, y, 0)}{((x - x_0)^2 + (y - y_0)^2 + z_0^2)^{\frac{3}{2}}} dx dy\,. \end{aligned}$$

13 Wave equation in higher dimensions

After focussing on elliptic pdes for the last two sections, we now turn our attention back to waves, although in higher dimensions. The wave equation in three dimensions is given by

$$u_{tt} - c^2 \Delta u = u_{tt} - c^2 \left(u_{xx} + u_{yy} + u_{zz}\right) = 0\,,$$

where c is the wave speed. Let us revisit the solution to the wave equation in one dimension. The solution to

$$u_{tt} = c^2 u_{xx}\,,$$

with initial conditions

$$u(x, 0) = \phi(x)\,, \quad \partial_t u(x, 0) = \psi(x)\,,$$

is given by

$$u(x, t) = \frac{1}{2}\left(\phi(x + ct) + \phi(x - ct)\right) + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds\,.$$

Some properties of the wave equation were:

- Conservation of energy: the energy

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} \left(\partial_t u(x, t)\right)^2 + c^2 \left(\partial_x u\right)^2 dx$$

  remains constant if u satisfies the wave equation. This gives a straightforward proof of uniqueness of solutions to the wave equation.

- Information cannot travel faster than speed c, which is also summarized in the characteristics being $x \pm ct$, also by the fact that the domain of dependence at the point (x_0, t_0) is given by the interval $(x_0 - ct_0, x_0 + ct_0)$, i.e. if the initial data ϕ and ψ are 0 on the interval $(x_0 - ct_0, x_0 + ct_0)$, then the solution to the wave equation is 0 everywhere in the light cone
$$\{(x,t) : x - x_0 - c(t-t_0) < 0\,, \quad x - x_0 + c(t-t_0) > 0\}\,,$$
which is furthermore reflected in the formula for the solution.

Turns out, analogous properies are also true for solutions to the wave equation in higher dimensions. The domain of dependence at the point $(\boldsymbol{x}_0, t_0)$ is the light cone

$$|\boldsymbol{x} - \boldsymbol{x}_0| \leq c|t - t_0|\,,$$

and the principle of causality also holds. The analogous energy

$$E(t) = \frac{1}{2}\iint \left(u_t^2 + c^2|\nabla u|^2\right) d\boldsymbol{x}\,,$$

is also conserved.

13.1 Conservation of energy and uniqueness

We present here a formal proof of the conservation of energy without worrying about convergence issues or switching orders of limiting processes.

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}\frac{1}{2}\iiint u_t^2 + c^2|\nabla u|^2 d\boldsymbol{x} \\ &= \frac{1}{2}\iiint \partial_t\left(u_t^2 + c^2|\nabla u|^2\right) d\boldsymbol{x} \\ &= \iiint \left(u_t u_{tt} + c^2 \nabla u_t \cdot \nabla u\right) d\boldsymbol{x}\end{aligned}$$

If

$$|\nabla u(\boldsymbol{x},t)| + |\partial_t u(\boldsymbol{x},t)| \leq \frac{M}{|\boldsymbol{x}|}\,,$$

for sufficiently large $\boldsymbol{x}$ and every t, then using the divergence theorem

$$0 = \iiint \nabla\cdot(u_t \nabla u) d\boldsymbol{x} = \iiint \left(u_t \Delta u + \nabla u_t \cdot \nabla u\right) d\boldsymbol{x}\,.$$

Combining the above two equations, we get

$$\frac{dE}{dt} = \iiint \left(u_t u_{tt} - c^2 u_t \Delta u\right) d\boldsymbol{x} = 0\,.$$

Thus, the energy is conserved. This gives a straight forward proof of uniqueness as well. If $\phi = \psi \equiv 0$, then $E(0) = 0$. Since $dE/dt = 0$, we conclude that $E(t) \equiv 0$.

$$0 = E(t) = \frac{1}{2}\iiint \left(u_t^2 + c^2|\nabla u|^2\right) d\boldsymbol{x} \implies u_t = 0\,, \quad \nabla u = \mathbf{0}\,.$$

Since the gradient of u is zero, we conclude that u is a constant and since $u = 0$ at $t = 0$, we further conclude that $u \equiv 0$.

13.2 Principle of causality

The principle of causality states that if ϕ and ψ are zero in the domain $|\boldsymbol{x} - \boldsymbol{x}_0| < ct_0$, then u is zero in the light cone $\{(\boldsymbol{x},t) : |\boldsymbol{x} - \boldsymbol{x}_0| < c(t_0 - t)\}$. To prove this result, let us fix a time slice t' and consider the frustrum

$$F = \{(\boldsymbol{x},t) : |\boldsymbol{x} - \boldsymbol{x}_0| < c(t_0 - t)\,, \quad t < t'\}\,.$$

The boundary of this light cone consists of three components $\partial F = T \cup B \cup K$ where

$$B = \{(\boldsymbol{x},0) : |\boldsymbol{x} - \boldsymbol{x}_0| < ct_0\}\,, \quad T = \{(\boldsymbol{x},t') : |\boldsymbol{x} - \boldsymbol{x}_0| < c(t_0 - t')\}\,, \quad K = \{(\boldsymbol{x},t) : |\boldsymbol{x} - \boldsymbol{x}_0| = c(t_0 - t) \quad t < t'\}\,.$$

Following the proof in the conservation of energy, we first observe that

$$0 = u_t\left(u_{tt} - c^2\Delta u\right) = \partial_t\left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right) + \partial_x\left(-c^2 u_t u_x\right) + \partial_y\left(-c^2 u_t u_y\right) + \partial_z\left(-c^2 u_t u_z\right).$$

Using the four dimensional divergence theorem, we get

$$\iiint_F \nabla\cdot\left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2, -c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z\right)dV = \iiint_{\partial F}\left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2, -c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z\right)\cdot \boldsymbol{n}\, dS$$

As discussed earlier, $\partial F = T \cup B \cup K$. Suppose the normal vector is given by $\boldsymbol{n} = (n_t, n_x, n_y, n_z)$ On the bottom surface B, the normal vector is given by

$$\boldsymbol{n}|_{\text{Bottom}} = (-1, 0, 0, 0).$$

On the top surface T, the normal vector is given by

$$\boldsymbol{n}|_{\text{Top}} = (1, 0, 0, 0).$$

And on the conical surface, the unit normal is given by

$$\boldsymbol{n}|_{\text{Cone}} = \frac{c}{\sqrt{c^2+1}}\left(1, \frac{x-x_0}{cr}, \frac{y-y_0}{cr}, \frac{z-z_0}{cr}\right),$$

where $r = c(t_0 - t)$. The normal on the conical surface is computed using the standard procedure. The surface is given by the level sets of the function $f(\boldsymbol{x}, t) = |\boldsymbol{x} - \boldsymbol{x}_0| - c(t_0 - t)$, the normal to which is given by $\nabla f/|\nabla f|$. Plugging in all the normals in the four dimensional divergence theorem, we get

$$\begin{aligned}\iiint_T \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2, -c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z\right)\cdot \boldsymbol{n}\, dS &= \iiint_T \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2, -c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z\right)\cdot(1,0,0,0)\, dS \\ &= \iiint_{|\boldsymbol{x}-\boldsymbol{x}_0| = c(t_0 - t')}\left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right)dS\end{aligned}$$

$$\begin{aligned}\iiint_B \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2, -c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z\right)\cdot \boldsymbol{n}\, dS &= \iiint_B \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2, -c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z\right)\cdot(-1,0,0,0)\, dS \\ &= -\iiint_{|\boldsymbol{x}-\boldsymbol{x}_0| = c(t_0)}\left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right)dS\end{aligned}$$

$$\begin{aligned}&\iiint_K \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2, -c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z\right)\cdot \boldsymbol{n}\, dS \\ &= \frac{c}{\sqrt{c^2+1}}\iiint_K \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2, -c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z\right)\cdot(1, \frac{x-x_0}{cr}, \frac{y-y_0}{cr}, \frac{z-z_0}{cr})\, dS \\ &= \frac{c}{\sqrt{c^2+1}}\iiint_{|\boldsymbol{x}-\boldsymbol{x}_0| = c(t_0-t)}\left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2\right) - cu_t\nabla u\cdot\hat{r}\, dS \\ &= \frac{c}{\sqrt{c^2+1}}\iiint_{|\boldsymbol{x}-\boldsymbol{x}_0| = c(t_0-t)}\left(\frac{1}{2}\left(u_t - cu_r\right)^2 + \frac{1}{2}c^2\left(|\nabla u - u_r\hat{r}|^2\right)\right)dS \geq 0.\end{aligned}$$

Thus,

$$\begin{aligned}\iiint_{\partial F}\left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2, -c^2u_tu_x, -c^2u_tu_y, -c^2u_tu_z\right)\cdot \boldsymbol{n}\, dS &= \iiint_{|\boldsymbol{x}-\boldsymbol{x}_0| = c(t_0-t')}\left(\frac{1}{2}u_t^2(t', \boldsymbol{x}) + \frac{1}{2}c^2|\nabla u(t', \boldsymbol{x})|^2\right)dS - \\ &\iiint_{|\boldsymbol{x}-\boldsymbol{x}_0| = ct_0}\left(\frac{1}{2}u_t^2(0, \boldsymbol{x}) + \frac{1}{2}c^2|\nabla u(0, \boldsymbol{x})|^2\right)dS + \iiint_K (\cdot)\, dS = 0\end{aligned}$$

$$\therefore \iiint_{|\boldsymbol{x}-\boldsymbol{x}_0| = ct_0}\left(\frac{1}{2}u_t^2(0, \boldsymbol{x}) + \frac{1}{2}c^2|\nabla u(0, \boldsymbol{x})|^2\right)dS \geq \iiint_{|\boldsymbol{x}-\boldsymbol{x}_0| = c(t_0-t')}\left(\frac{1}{2}u_t^2(t', \boldsymbol{x}) + \frac{1}{2}c^2|\nabla u(t', \boldsymbol{x})|^2\right)dS$$

where the last inequality follows from the fact that the integral over the conical surface K of the flux is positive. The above result shows that if u and u_t are zero in $|\boldsymbol{x} - \boldsymbol{x}_0| \leq ct_0$, then $u_t(t', \boldsymbol{x})$ and $\nabla u(t', \boldsymbol{x})$ are zero in $|\boldsymbol{x} - \boldsymbol{x}_0| \leq c(t_0 - t')$ and hence u is a constant in the light cone. since $u = 0$ at $t = 0$, we conclude that u is 0 in the light cone everywhere.

13.3 Solution in two and three dimensions

We now present formulae for the solution to the wave equation in two and three dimensions for given initial conditions, $\phi(\boldsymbol{x})$ and $\psi(\boldsymbol{x})$.

Theorem 13.1. *In three dimensions, if u satisfies*

$$u_{tt} = c^2 \Delta u\,,$$

with initial conditions

$$u(0,\boldsymbol{x}) = \phi(\boldsymbol{x})\,, \quad \text{and} \quad \partial_t u(0,\boldsymbol{x}) = \psi(\boldsymbol{x})\,,$$

then

$$u(t_0,\boldsymbol{x}_0) = \frac{1}{4\pi c^2 t_0}\iint_{|\boldsymbol{x}-\boldsymbol{x}_0|=ct_0} \psi(\boldsymbol{x})dS + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\iint_{|\boldsymbol{x}-\boldsymbol{x}_0|=ct} \phi(\boldsymbol{x})dS\right)|_{t=t_0}\,.$$

Remark 13.1. *The second term in the above expression should be interpreted as follows: for a fixed $\boldsymbol{x}_0$ consider the following function of t*

$$f(t) = \frac{1}{4\pi c^2 t}\iint_{|\boldsymbol{x}-\boldsymbol{x}_0|=ct} \phi(\boldsymbol{x})\,dS\,.$$

Then the second term is given by

$$\frac{df}{dt}(t_0)\,.$$

To prove the above result, we use the method of spherical means. Without loss of generality, we set $\boldsymbol{x}_0 = 0$. Otherwise we may consider $v(\boldsymbol{x}) = u(\boldsymbol{x}+\boldsymbol{x}_0)$. Let

$$\overline{u}(t,r) = \frac{1}{4\pi r^2}\iint_{\partial B_r(0)} u(t,\boldsymbol{x})dS\,.$$

$\overline{u}(t,r)$ represents the average of the values of u on the boundary of the sphere of radius r centered at the origin. It can be shown that the spherical mean of u itself satisfies the one dimensional wave equation in polar coordinates. Since u satisfies the wave equation, u is twice differntiable and we can switch the order of differentiation and integration in the above equation to conclude that

$$\partial_{tt}\overline{u}(t,r) = \partial_{tt}\left(\frac{1}{4\pi r^2}\iint_{\partial B_r(0)} u(t,\boldsymbol{x})\,dS\right) = \frac{1}{4\pi r^2}\iint_{\partial B_r(0)} \partial_{tt}u(t,\boldsymbol{x})\,dS\,.$$

We will now show that $\Delta\overline{u}(t,r) = \overline{\Delta u}$, for which we need to use the Laplacian in spherical coordinates.

$$\begin{aligned}
\overline{\Delta u} &= \frac{1}{4\pi r^2}\iint_{\partial B_r(0)} \Delta u(t,r,\theta,\phi)dS \\
&= \frac{1}{4\pi r^2}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left[\left(\Delta_r + \frac{1}{r^2\sin(\theta)^2}\partial_{\phi\phi} + \frac{1}{r^2\sin(\theta)}(\cos(\theta)\partial_\theta + \sin(\theta)\partial_{\theta\theta})\right)u(t,r,\theta,\phi)\right]r^2\sin(\theta)d\theta d\phi\,,
\end{aligned}$$

where Δ_r is the radial contribution of the laplacian which in three dimensions is given by

$$\Delta_r = \partial_{rr} + \frac{2}{r}\partial_r\,.$$

Note that in spherical coordinates, for each fixed θ and r, $u(t,r,\theta,\phi)$ is a 2π periodic function in ϕ. Thus,

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{1}{r^2\sin(\theta)^2}\partial_{\phi\phi}u(t,r,\theta,\phi)r^2\sin(\theta)d\theta d\phi = 0$$

We now turn our attention to the θ terms and use integration by parts for $\cos(\theta)\partial_\theta u$ term.

$$\begin{aligned}
\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{1}{r^2\sin(\theta)}(\cos(\theta)\partial_\theta u + \sin(\theta)\partial_{\theta\theta}u)r^2\sin(\theta)d\theta d\phi &= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\cos(\theta)u_\theta + \sin(\theta)u_{\theta\theta}d\theta d\phi \\
&= \int_{\phi=0}^{2\pi}\left[\left(\sin(\theta)u_\theta|_{\theta=0}^{\theta=\pi} - \int_{\theta=0}^{\pi}\sin(\theta)u_{\theta\theta}d\theta\right) + \int_{\theta=0}^{\pi}\sin(\theta)u_{\theta\theta}d\theta\right]d\phi = 0
\end{aligned}$$

Thus,

$$\overline{\Delta u} = \frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\Delta_r u(t,r,\theta,\phi)\sin(\theta)d\theta d\phi = \Delta_r\left[\frac{1}{4\pi}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}u(t,r,\theta,\phi)\sin(\theta)d\theta d\phi\right] = \Delta_r\overline{u}(t,r)\,.$$

Since u satisfies the wave equation, we have

$$\partial_{tt}u = c^2 \Delta u \,,$$
$$\overline{\partial_{tt}u} = c^2 \overline{\Delta u} \,,$$
$$\partial_{tt}\overline{u}(t,r) = c^2 \Delta_r \overline{u}(t,r) \ ,.$$

Now suppose $v(t,r) = r\overline{u}(t,r)$, then

$$\partial_{rr}v = r\Delta_r \overline{u}(t,r) \,, \quad \partial_{tt}v = r\partial_{tt}\overline{u} \,.$$

Thus, v satisfies the one dimensional wave equation

$$\partial_{tt}v(t,r) = c^2 \partial_{rr}v(t,r) \,,$$

for $r > 0$ with boundary conditions.

$$v(t,0) = 0 \cdot \overline{u}(t,0) = 0 \,,$$

and initial conditions

$$v(0,r) = r\overline{u}(0,r) = r\frac{1}{4\pi r^2} \iint_{\partial B_r(0)} u(0,\boldsymbol{x})dS = r\frac{1}{4\pi r^2} \iint_{\partial B_r(0)} \phi(\boldsymbol{x})dS = r\overline{\phi}(r) \,,$$

and

$$\partial_t v(0,r) = r\partial_t \overline{u}(0,r) = r\partial_t \frac{1}{4\pi r^2} \iint_{\partial B_r(0)} u(0,\boldsymbol{x})dS = r\frac{1}{4\pi r^2} \iint_{\partial B_r(0)} \partial_t u(0,\boldsymbol{x})dS = r\frac{1}{4\pi r^2} \iint_{\partial B_r(0)} \psi(\boldsymbol{x})dS = r\overline{\psi}(r) \,.$$

From the solution to one dimensional wave equation on the half line $0 < r < \infty$ with 0 dirichlet boundary data, $v(t,r)$ is given by

$$\begin{aligned} v(t,r) &= \frac{1}{2c}\int_{ct-r}^{ct+r} s\overline{\psi}(s)\,ds + \frac{1}{2}\big((r+ct)\overline{\phi}(r+ct) + (ct-r)\overline{\phi}(ct-r)\big) \\ &= \frac{1}{2c}\int_{ct-r}^{ct+r} s\overline{\psi}(s)\,ds + \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{ct-r}^{ct+r} s\overline{\phi}(s)\,ds\right] \,, \end{aligned}$$

for $0 \leq r \leq ct$ and a different expression for $r > ct$. Stepping back $r\overline{u}(t,r) = v(t,r)$ and $u(t,0) = \lim_{r\to 0}\overline{u}(t,r) = \overline{u}(t,0)$. Thus, we only care about v for $0 \leq r \leq ct$.

$$\overline{u}(t,0) = \lim_{r\to 0}\frac{v(t,r)}{r} = \lim_{r\to 0}\frac{v(t,r) - v(t,0)}{r} = \partial_r v(t,0) \,,$$

since $v(t,0) = 0$.

$$\partial_r v(t,r) = \frac{1}{2c}\left[(ct+r)\overline{\psi}(ct+r) + (ct-r)\overline{\psi}(ct-r)\right] + \ldots$$

and

$$u(t,0) = \overline{u}(t,0) = t\overline{\psi}(ct) + \ldots = \frac{1}{4\pi c^2 t}\iint_{|\boldsymbol{x}|=ct} \psi(\boldsymbol{x})dS + \ldots \,,$$

which completes the proof.

The above formula also demonstrates how waves in three dimensions are different from waves in one and two dimensions (which we shall see soon). For waves in one dimension, the domain of dependence for the solution at (t_0, x_0) is the whole interval $|x - x_0| \leq ct_0$, however, for waves in three dimensions, the domain of dependence is only the boundary of the sphere $|\boldsymbol{x} - \boldsymbol{x}_0| = ct_0$. This tells us that waves travel on wave fronts in three dimensions but not in one dimension. If the initial data corresponded to a ping at the origin in 1-dimension and you were observing the solution at a fixed point in space x_0 as a function of time, then you would hear the ping forever after time $t_0 = x_0/c$. However, in three dimensions, for a ping of time width δ, the ping would be audible for a time interval

$$t \in \left[|\boldsymbol{x}_0|/c, |\boldsymbol{x}_0|/c + \delta\right] \,.$$

This principle is referred to as Huygen's principle - waves travel on wavefronts and the solution propogates at exactly the speed c, no faster, no slower.

13.4 Solution in two dimensions

One could construct the solution to the wave equation using the method of spherical means in two dimensions as well, but the PDE corresponding to $v = r \cdot \overline{u}(t,r)$ would not be the wave equation and would require a different solution procedure. Instead of using the method of spherical means, we use the solution in three dimensions to construct the solution in two dimensions by what is referred to as the method of descent.

Consider the wave equation in two dimensions,

$$u_{tt} = c^2(u_{xx} + u_{yy}) \,, \quad u(0,x,y) = \phi(x,y) \,, \quad \text{and} \quad u_t(0,x,y) = \psi(x,y) \,.$$

We think of u as a solution to the wave equation in three dimensions which does not depend on z, i.e. set $v(t,x,y,z) = u(t,x,y)$, if you like and clearly v satisfies the wave equation in three dimensions with initial data same as u. Using the procedure discussed in the previous section, the solution to v at $0,0,0$ is given by

$$v(t,0,0,0) = u(t,0,0) = \frac{1}{4\pi c^2 t} \iint_{x^2+y^2+z^2=c^2t^2} \psi(x,y) dS + \ldots$$

Since the integrand in the above expression does not depend on z, the integral over $z = \sqrt{(c^2t^2 - x^2 - y^2)}$ is exactly the same as the integral over $z = -\sqrt{(c^2t^2 - x^2 - y^2)}$ using symmetry. Furthermore, the surface are element for the surface $z(x,y) = \sqrt{(c^2t^2 - x^2 - y^2)}$ is given by

$$dS = \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right]^{\frac{1}{2}} dxdy = \left[1 + \left(\frac{-x}{z}\right)^2 + \left(\frac{-y}{z}\right)^2\right]^{\frac{1}{2}} dxdy = \frac{ct}{z} dxdy = \frac{ct}{\sqrt{(c^2t^2 - x^2 - y^2)}} dxdy \,.$$

Thus,

$$\begin{aligned} u(t,0,0) &= \frac{1}{2\pi c^2 t} \iint_{z=\sqrt{(c^2t^2-x^2-y^2}} \psi(x,y) dS + \ldots \\ &= \frac{1}{2\pi c^2 t} \iint_{x^2+y^2 \leq c^2t^2} \psi(x,y) \left[1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2\right]^{\frac{1}{2}} dxdy + \ldots \\ &= \frac{1}{2\pi c} \iint_{x^2+y^2 \leq c^2t^2} \frac{\psi(x,y)}{\sqrt{(c^2t^2 - x^2 - y^2)}} dxdy + \ldots . \end{aligned}$$

For a general point the formula is given by

$$u(t,x_0,y_0) = \frac{1}{2\pi c} \iint_{x^2+y^2 \leq c^2t_0^2} \frac{\psi(x,y)}{\sqrt{(c^2t_0^2 - (x-x_0)^2 + (y-y_0)^2)}} + \frac{\partial}{\partial t}\left(\frac{1}{2\pi c} \iint_{x^2+y^2 \leq c^2t^2} \frac{\phi(x,y)}{\sqrt{(c^2t^2 - (x-x_0)^2 + (y-y_0)^2)}} + \right)|_{t=t_0} \,.$$

From the formula above, we see that waves in two dimensions do not obey Huygen's principle, i.e. the domain of dependence for the point $(t_0, \boldsymbol{x}_0)$ is the whole light cone $|\boldsymbol{x} - \boldsymbol{x}_0| \leq ct_0$. Thus, if you have initial data supported on a small region around the origin, and you observe the solution at a fixed location $\boldsymbol{x}_0$ as a function of time t, then the solution is always non zero after time $t = |\boldsymbol{x}_0|/c$ and decays as $\frac{1}{\sqrt{t}}$ as $t \to \infty$. You can observe this phenomenon in day to day practice when you throw a pebble in a steady lake for example, the ripples continue to survive for a long time. However in reality, you have damping or friction or dissipation which eventually sends the energy to 0.

14 Calculus of variations

We have already seen the connection between minimization of an energy functional and solution to Laplace's equation. Solutions to Laplace's equation with Dirichlet boundary conditions minimize the Dirichlet energy

$$E[w] = \iint_D |\nabla w|^2 \, dV$$

where w lies in an appropriate family of functions. The theory is a lot more general than that. Many phyical priniciples are known to us as minimizers of a certain action. Just like we did with the Dirichlet energy, we can convert the minimization problem into a PDE.

For example, consider the path travelled by a ray of light in an inhomogeneous medium. Suppose the speed of light in the medium is given by $c(x,y) = c(\boldsymbol{x})$. Then according to Feynman, the path taken by a ray of light to travel from (a, y_a) to

(b, y_b) is the one which takes minimum time. Suppose the ray of light travels along the curve $(x, u(x))$. Then the infinitesimal time taken by light to travel a small distance dx is given by

$$\frac{\sqrt{\left(1 + (\frac{du}{dx})^2\right)}\, dx}{c(x, u(x))}$$

Then the path $u(x)$ taken by the ray of light is the one which minimizes the functional

$$T[u] = \int_a^b \frac{1}{c(x, u(x))} \sqrt{\left(1 + \left(\frac{du}{dx}\right)^2\right)}\, dx$$

subject to the constraint $u(a) = y_a$ and $u(b) = y_b$. For every function $u(x)$, we get a number corresponding to the time taken by the ray of light to travel along the curve. The goal is to find the curve for which that time taken is minimum. It takes some effort to show that the time functional indeed has a minimum and that a curve $u(x)$ does exist which achieves this minimum value. However, it can be shown that if such a curve does exist, then it must also satisfy an ordinary differential equation. For this let us consider the problem in abstraction and then we will come back to the example. Suppose we wish to minimize the functional

$$\mathcal{F}[u] = \int_a^b F(x, u, u'(x))dx\,,$$

subject to the constraint $u(a) = y_a$ and $u(b) = y_b$. If such a function u does exist for which the above functional is minimized, then the function of one dimenion

$$f(t) = \mathcal{F}[u + tv] = \int_a^b F(x, u + tv, u' + tv')\, dx\,,$$

achieves it's minimum at $t = 0$ whenever $v(a) = v(b) = 0$. The boundary conditions are required to ensure that $u + tv$ remains in the admissible class of functions in which the functional $\mathcal{F}$ has achieved it's minimum value. Since f achieves it's minimum at $t = 0$, we get to say that

$$f'(0) = 0\,.$$

Using the chain rule, the derivative of f is given by

$$f'(t) = \frac{d}{dt}\int_a^b F(x, u{+}tv, u'{+}tv')\, dx = \int_a^b \frac{d}{dt}\left(F(x, u + tv, u' + tv')\right) dx = \int_a^b \left((\partial_2 F(x, u + tv, u' + tv')) \cdot v + (\partial_3 F(x, u + tv, u' + tv')) \cdot v'\right) dx\,,$$

where $\partial_2 F(x, y, z)$ is the partial derivative of F where we vary the second argument y and hold the first and the third arguments x, z fixed, i.e.

$$\partial_2 F(x, y, z) = \lim_{h\to 0} \frac{F(x, y + h, z) - F(x, y, z)}{h}\,.$$

Similarly,

$$\partial_3 F(x, y, z) = \lim_{h\to 0} \frac{F(x, y, z + h) - F(x, y, z)}{h}\,.$$

We turn our attention back to f'

$$\begin{aligned}
0 = f'(0) &= \int_a^b \left((\partial_2 F(x, u(x), u'(x))) \cdot v(x) + (\partial_3 F(x, u(x), u'(x))) \cdot v'(x)\right) dx\,,\\
&= \int_a^b v(x) \cdot \partial_2 F(x, u(x), u'(x)) + v(x) \cdot \partial_3 F(x, u(x), u'(x))|_{x=a}^b - \int_a^b v(x) \cdot \frac{d}{dx}\left(\partial_3 F(x, u(x), u'(x))\right) dx \quad \text{(integration by parts)}\\
&= \int_a^b v(x) \cdot \left(\partial_2 F(x, u(x), u'(x)) - \frac{d}{dx}\left(\partial_3 F(x, u(x), u'(x))\right)\right) dx + v(b)\partial_3 F(b, u(b), u'(b)) - v(a)\partial_3 F(a, u(a), u'(a))\\
&= \int_a^b v(x) \cdot \left(\partial_2 F(x, u(x), u'(x)) - \frac{d}{dx}\left(\partial_3 F(x, u(x), u'(x))\right)\right) dx \quad \text{(since } v(a) = v(b) = 0)\,.
\end{aligned}$$

The above equation holds for all v with $v(a) = v(b) = 0$, and thus we must have

$$\partial_2 F(x, u(x), u'(x)) = \frac{d}{dx}\left(\partial_3 F(x, u(x), u'(x)\right) \qquad a < x < b\,.$$

Thus if u is the minimizer of the functional $\mathcal{F}$, then u must satisfy the above differential equation. In our example,

$$F(x, y, z) = \frac{1}{c(x, y)}\sqrt{(1 + z^2)}\,,$$

$$\partial_2 F(x,y,z) = -\frac{\partial_2 c(x,y)}{c^2(x,y)}\sqrt{(1+z^2)}\,,$$

and

$$\partial_3 F(x,y,z) = \frac{1}{c(x,y)}\frac{z}{\sqrt{(1+z^2)}}\,.$$

Then u must satisfy the ODE,

$$-\frac{\partial_2 c(x,u(x))}{c^2(x,u(x))}\sqrt{(1+u'(x)^2)} = \frac{d}{dx}\left(\frac{1}{c(x,u(x))}\frac{u'(x)}{\sqrt{(1+u'(x)^2}}\right)\,.$$

Similarly, in higher dimensions, we can derive PDEs for the minimizer of functionals. For example, consider the shape of the surface assumed by a wire frame immersed in a soap solution. The physical principle at work here is that the shape assumed by the wire is one which minimizes surface area. However, the boundary of the wire frame, imposes a restriction on the boundary of the surface. Let $z(x,y)$ be a parametrization of the surface. Suppose the boundary of the wire frame denoted by ∂D is given by

$$z(x,y) = h(x,y) \quad (x,y)\in\partial D\,. \tag{95}$$

Let D denote the interior of the wireframe projected onto the $x-y$ plane. The corresponding surface area is

$$A[z] = \iint_D \sqrt{\left(1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right)}\,dV\,.$$

Given any surface z we can compute the area of the surface, and the surface assumed by the soap film will be the one which minimized the above functional subject to the constraint given by equation (95).

If such a minimizer does exist, we can proceed as before to derive a PDE for the minimizing surface u by observing that

$$f(t) = A[u+tv] = \iint_D F(u+tv, u_x+tv_x, u_y+tv_y)dV$$

achieves it's minimum value at $t=0$ for any function v which is 0 on the boundary $(x,y)\in\partial D$ and hence f must satisfy $f'(0)=0$. Here

$$F(x,y,z) = \sqrt{1+y^2+z^2}\,.$$

It follows from a simple calculation similar to the one dimensional case and using the divergence theorem instead of integration by parts that

$$0 = f'(0) = \iint_D \left(v\cdot\left(\partial_1 F(u,u_x,u_y) - \frac{\partial}{\partial x}\left[\partial_2 F(u,u_x,u_y)\right] - \frac{\partial}{\partial y}\left[\partial_3 F(u,u_x,u_y)\right]\right)\right)dV\,.$$

Since the above equation holds for all functions v which are 0 on the boundary ∂D, we conclude that the minimizer u must satisfy the PDE

$$\partial_1 F(x,u_x,u_y) = \frac{\partial}{\partial x}\left[\partial_2 F(u,u_x,u_y)\right] + \frac{\partial}{\partial y}\left[\partial_3 F(u,u_x,u_y)\right]\,.$$

For the example of the soap film, the PDE is given by

$$\frac{\partial}{\partial x}\left[\frac{u_x}{\sqrt{1+u_x^2+u_y^2}}\right] + \frac{\partial}{\partial y}\left[\frac{u_y}{\sqrt{1+u_x^2+u_y^2}}\right] = 0\,.$$

Indeed in the limit of small gradients, i.e. u_x and u_y being small, the "energy" being minimized is the dirichlet energy

$$A[z] = \iint_D \sqrt{1+z_x^2+z_y^2}dV \approx \iint_D (1+\frac{1}{2}\left(z_x^2+z_y^2\right)\,dV\,,$$

since $\sqrt{1+x} = 1+\frac{x}{2}$ for small x and the corresponding PDE we have derived becomes Laplace's equation, since

$$0 = \frac{\partial}{\partial x}\left[\frac{u_x}{\sqrt{1+u_x^2+u_y^2}}\right] + \frac{\partial}{\partial y}\left[\frac{u_y}{\sqrt{1+u_x^2+u_y^2}}\right] = \frac{\partial}{\partial x}u_x + \frac{\partial}{\partial y}u_y\,.$$

where we have assumed $\sqrt{1+u_x^2+u_y^2}\approx 1$.

14.1 Nonlinear equations and shock waves

We now turn our attention to the simplest first order non-linear PDE, *Burgers equation*, given by

$$u_t + uu_x = 0 \iff \partial_t u + \partial_x A(u) = 0\,, \tag{96}$$

where $A(u) = u^2/2$. We wish to solve the above PDE subject to the initial condition

$$u(0,x) = \phi(x)\,.$$

Since this is a first order equation, we can attempt to use the method of characteristics to construct a solution to this differential equation. The characteristics for the above PDE are given by

$$\frac{dx}{dt} = u(t,x)\,. \tag{97}$$

Thus, the characteristics have slope corresponding to the unknown solution $u(t,x)$ that we are seeking. However, suppose we computed the solution using some other mechanism, and constructed the constructed the corresponding characteristics $x(s)$ which satisfy the ODE (97). Then on the curve $(s, x(s))$, u satisfies the differential equation

$$\frac{d}{ds}u(s,x(s)) = \partial_t u(s,x(s)) + \partial_x u(s,x(s))\frac{dx(s)}{ds} = \partial_t u(s,x(s)) + u(s,x(s))\partial_x u(s,x(s)) = 0\,.$$

Thus, along the characteristics $(s, x(s))$, the solution $u(s, x(s))$ is a constant. This implies that the characteristics for (96) are still straightlines with slope $\phi(x_0)$ passing through the point $(0, x_0)$. The equation of the characteristics are

$$x - x_0 = \phi(x_0)t\,.$$

The above procedure gives us a solution to the problem for $t > 0$ as long as none of the characteristics intersect or given a point (t, x) we can find a characteristic passing through it. Neither of them would necessarily be the case as is readily observed by using the initial data

$$\phi_s(x) = \begin{cases} 1 & x < 0 \\ 0 & x \geq 0 \end{cases}$$

$$\phi_r(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

For the initial data ϕ_s the characteristics for $x < 0$ end up interesecting the characteristics at $x \geq 0$ for any time $t > 0$. And for the initial data ϕ_r, there are no characteristics in the triangular region $0 < x < t$ and $t \geq 0$.

Before we address this issue in detail, let us look at a couple of examples. If the funtion $\phi(x)$ is monotonically increasing, then characteristics do not intersect and we can always compute the solution. For example, consider the initial data:

$$\phi(x) = x + 2\,.$$

The characteristics are given by

$$x - x_0 = \phi(x_0)t\,, \quad x_0(t+1) = x - 2t\,.$$

Thus, the solution at (t, x) is given by

$$u(t,x) = \phi(x_0) = \phi\left(\frac{x-2t}{t+1}\right) = \frac{x-2t}{t+1} + 2 = \frac{x+2}{t+1}\,.$$

It is easy to verify that u above does satisfy the PDE and has the right initial data.

$$\partial_t u(t,x) = -\frac{x+2}{(t+1)^2} \qquad \partial_x u(t,x) = \frac{1}{t+1}$$

$$\partial_t u + u\partial_x u = -\frac{x+2}{(t+1)^2} + \frac{x+2}{t+1}\cdot\frac{1}{t+1} = 0\,.$$

Thus,u does infact solve the PDE. Furthermore,

$$u(0,x) = \frac{x+2}{(0+1)} = x + 2\,.$$

Now consider the example where $\phi(x) = x^2$. The charactersistics in this case are given by

$$x - x_0 = \phi(x_0)t \quad x_0^2 t + x_0 - x = 0\,.$$

Thus, given (t,x), we can solve the above equation to obtain $x_0(t,x)$ as a function of (t,x) and then the solution at (t,x) is given by $\phi(x_0(t,x))$. The argument breaks down when $x_0(t,x)$ stops being a function, i.e. either it becomes multivalued or there does not exist a solution for the point (t,x). For our currenty example, we have two options

$$x_0(t,x) = \frac{-1 \pm \sqrt{1+4tx}}{2t}\,.$$

Firstly, we need to choose the branch of the square root which in the limit $t \to 0$ gives us $x_0 = x$. Clearly, the branch corresponding to $\frac{-1-\sqrt{1+4tx}}{2t}$ converges to -∞ in the limit $t \to 0$. On the other hand,

$$\lim_{t\to 0} \frac{-1+\sqrt{1+4tx}}{2t} = \frac{1}{2}\cdot\frac{4x}{2\sqrt{1+4tx}} = x$$

Thus, we must choose

$$x_0 = \frac{-1+\sqrt{1+4tx}}{2t}\,,$$

and the solution $u(t,x)$ is given by

$$u(t,x) = \phi(x_0(t,x)) = x_0^2 = \frac{1+2tx-\sqrt{1+4tx}}{2t^2}\,.$$

The above solution is fine as long as $4tx > -1$. However, when $x < 0$, there exists t_0 beyond which the above classical solution ceases to exist.

So it is clear from the above discussion, that we may not necessarily have smooth solutions to the PDE even when the initial data is also smooth. So can we weaken our definition for what it means to be solution and look for non-smooth solutions instead, that satisfy the PDE in the classical sense, i.e. pointwise, as in the first example for most of the domain except for allowing for discontinuities on curves.

We will refer to a solution which satisfies the PDE pointwise and is C^1 in all of (t,x) space as a classical solution. For now let us assume that we have a classical solution $u(t,x)$. Let us multiply the PDE with a smooth solution $\psi(t,x)$ which is 0 outside of $|t| \leq t_0$ and $|x| \leq r_0$. We will refer to these functions as compactly supported functions in the (t,x) space.

$$0 = \int_{t=0}^{\infty}\int_{x=-\infty}^{\infty} \left(u_t(t,x) + uu_x(t,x)\right)\psi(t,x)dxdt\,.$$

The above equation is valid for all smooth $\psi(t,x)$. We can use integration parts by parts in the above equation to move the derivative from u to ψ.

$$0 = \int_{t=0}^{\infty}\int_{x=-\infty}^{\infty}\left(u_t(t,x)\psi(t,x) + \frac{1}{2}\partial_x u^2\psi(t,x)\right)dxdt = \int_{t=0}^{\infty}\int_{x=-\infty}^{\infty}\left(u(t,x)\psi_t(t,x) + \frac{1}{2}\psi_x(t,x)u^2(t,x)\right)dxdt \qquad (98)$$

The second equation clearly does not have any derivatives of the solution $u(t,x)$ involved. Any function $u(t,x)$ which satisfies the equation

$$\int_{t=0}^{\infty}\int_{x=-\infty}^{\infty}\left(u(t,x)\psi_t(t,x) + \frac{1}{2}\psi_x(t,x)u^2(t,x)\right)dxdt\,,$$

for all smooth compactly supported functions $\psi(t,x)$ will be referred to as a *weak solution* of the PDE

$$u_t + uu_x = 0\,.$$

Clearly, any classical solution of the PDE is a weak solution. It can be shown that any weak solution of the PDE which is also C^1 locally at the point (t,x) does infact satisfy the PDE pointwise as well by choosing an appropriate function ψ.

As noted before, weak solutions as defined above need not necessarily be C^1. Let us assume that we have a C^0 solution in the plane which is C^1 and satisfies the PDE pointwise everywhere in the (t,x) plane except for on a curve $\xi(t)$ across which u is also allowed to be discontinuous. The fact that u must be a weak solution, imposes certain restrictions on what the the curve $(t,\xi(t)$ along which the solution can be discontinuous. Let u^- denote the classical solution of the PDE for $(t,x) : \{x < \xi(t)\}$ and u^+ denote the classical solution to the right of the discontiuity $(t,x) : \{x > \xi(t)\}$. Now we split the inner integral over x on on the interval $-\infty$ to $\xi(t)$ and $\xi(t)$ to ∞. On each piece we apply the divergence theorem to get

$$\begin{aligned}
0 = \int_{t=0}^{\infty}\int_{x=-\infty}^{\xi(t)}\left[-\psi(t,x)u_t - \frac{1}{2}\psi(t,x)\partial_x u^2\right]dxdt - \int_{x=\xi(t)}\left[u^-\psi n_t + \frac{1}{2}\left(u^-\right)^2\psi n_x\right]dl \\
+ \int_{t=0}^{\infty}\int_{x=\xi(t)}^{\infty}[-\psi u_t - \frac{1}{2}\psi\partial_x u^2]dxdt \\
+ \int_{x=\xi(t)}\left[u^+\psi n_t + \frac{1}{2}\left(u^+\right)^2\psi n_x\right]dl\,.
\end{aligned}$$

where (n_t, n_x) denotes the unit normal vector to the shock vector that points to the right. Since u is a classical solution on either side of the curve $x = \xi(t)$, we conclude that the first and the third terms are 0 and what we are left with is

$$\int_{x=\xi(t)} \psi(t,x)\left[u^- n_t + \frac{1}{2}\left(u^-\right)^2 n_x\right] dl = \int_{x=\xi(t)} \psi(t,x)\left[u^+ n_t + \frac{1}{2}\left(u^+\right)^2 n_x\right] dl\,.$$

Since the above equation holds for all functions $\psi(t,x)$, we conclude that

$$\frac{A(u^+) - A(u^-)}{u^+ - u^-} = -\frac{n_t}{n_x} = \frac{d\xi}{dt}\,.$$

Thus, the slope of the discontinuity is related to the limiting values of the solution on either side and cannot just be arbitrary. This equation is referred to as the *Rankine-Hugoniot* conditions for determining the speed of the shock wave.

We turn our attention to the initial data ϕ_r and ϕ_s that we had before. For $\phi_s(x)$. the characteristics interesect. The limiting value on the left of where the characteristics intersect $u^- = 1$ and the limiting value on the right is $u^+ = 0$. Thus, the speed of the discontinuity along which solution is discontinuous

$$\frac{d\xi}{dt} = \frac{A(u^+) - A(u^-)}{u^+ - u^-} = \frac{0 - \frac{1}{2}}{0 - 1} = \frac{1}{2}$$

Thus, in this case, the shock wave is also a straight line in the (t,x) plane, given by

$$\xi(t) = t/2$$

Thus, the weak solution to

$$u_t + uu_x = 0\,, \quad u(0,x) = \phi_s(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases},$$

is given by

$$u(t,x) = \begin{cases} 1 & x < t/2 \\ 0 & x > t/2 \end{cases}.$$

Similarly, if we use the initial data

$$\phi_r(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

The speed of the discontinuity in this case is also given by

$$\xi(t) = t/2$$

and the solution $u(t,x)$ is then given by

$$u(t,x) = \begin{cases} 0 & x < t/2 \\ 1 & x > t/2 \end{cases}.$$

In this case, we filled the empty space between $0 < x < t$ and $t > 0$, by introducing a shock wave and then the characteristics emanate from the shock wave. An alternate solution to the PDE with initial data $\phi_r(x)$ is given by

$$u(t,x) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < t \\ 1 & x > t \end{cases}$$

It is a simple calculation to show that the above solution is also a weak solution to the PDE. This example shows that weak solutions to the Burgers equation are not unique. The question is which one is the physically correct solution and the solution that we'd observe. It's easy to argue that the continuous solution with x/t instead of the discontinuous solution but the argument is not physical. In physical systems there is typically dissipation even though it may be present in small amounts. So a closer physical model would be

$$u_t + uu_x = \varepsilon \Delta u$$

with $\varepsilon > 0$. However, mathematically it is advantageous to work in the limit $\varepsilon \to 0$ rather than trying to resolve the boundary layer associated the dissipation. It can be shown that weak solutions which are discontinuous must also satisfy the *entropy condition*

$$A(u^-) > \frac{d\xi}{dt} > A(u^+)$$

With some detailed analysis, it can be shown that there exists a unique weak entropic solution to Burgers equation. The moral of the story here is that the solution to

$$u_t + uu_x = 0\,, \quad u(0,x) = \phi_r(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}\,,$$

is given by

$$u(t,x) = \begin{cases} 0 & x < 0 \\ x/t & 0 < x < t \\ 1 & x > t \end{cases}\,.$$