Practice Problem set 4

April 2, 2018

1. Suppose that $X = c_0(\mathbb{N})$, then $X^* = \ell^1(\mathbb{N})$, and $X^{**} = \ell^\infty(\mathbb{N})$. Show that the unit ball of $c_0(\mathbb{N})$ is dense in $\ell^\infty(\mathbb{N})$ in the $\sigma(X^{**}, X^*)$ topology, i.e. for every element $f \in \ell^\infty$, there exists a sequence $g_n \in c_0(\mathbb{N})$ such that for every element $h \in \ell^1(\mathbb{N})$

$$(f - g_n, h) \to 0 \quad \text{as} \quad n \to \infty$$

2. Suppose that X is a Banach space. Use the uniform boundedness principle to show that any weak-$*$ convergent sequence in X^* is norm bounded. Use this to show that any weakly convergent sequence in X is also norm bounded.

3. Consider the definition of convergence in a topological vector space where $x_i \to x$ is for every neighborhood of U of x, x_i is in U eventually. Show that $x_i \to x$ weakly, if and only if $(x_i, x^*) \to (x_i, x^*)$ for each x^* in x_i. Show a similar result for weak-$*$ convergence.

4. Show that every Banach space X embeds isometrically into $C(K)$ for some compact topological space K. (Hint: use Banach Alaoglu)