Practice Problem set 3

February 26, 2018

1. Suppose that A is closed convex compact subset of X a complete vector space. Suppose that $F = \{x_0\}$ is an extreme set of A. Show that $x_0 \in \mathcal{E}(A)$.

2. Prove the inverse mapping theorem. If $T : X \to Y$ is linear, bounded and a bijection. Then show that T^{-1} is bounded.

3. Suppose that $T : X \to Y$ is linear and bounded. Show that T is an open map, i.e. $T(A)$ is open for every open set A, if and only if $T(B_1(0))$ contains a ball centered at the origin.

4. Suppose that B is a closed extreme subset of A, a compact closed convex set. Suppose that $\ell \in X^*$ is a bounded linear functional. Show that

$$F = \{x \in B \mid \ell(x) = \sup_{y \in B} \ell(y)\} ,$$

   is a non-empty, closed, convex extreme subset of A.

5. Suppose that $X = L^1[0,1]$. Show that $\overline{B_1(0)} \subset X$ has no extreme points.

6. Show that $\ell^p(\mathbb{N})$, $0 < p < 1$, is a metric space with the metric

$$d(a,b) = \sum_{j=1}^{\infty} |a_j - b_j|^p .$$

   Construct a compact set in X whose convex hull is unbounded.