Practice Problem set 2

February 13, 2018

1. Show that $(L^p[0,1])^* = L^q[0,1]$ for all $1 \leq p < \infty$ where $1/p + 1/q = 1$

2. If X is a normed space, show that X^* is a normed space

3. Suppose that $X = C([0,1])$ with the norm

$$\|f\| = \int_0^1 |f(t)| dt \,,$$

   and defined $L : X \to \mathcal{F}$ by

$$L(f) = f\left(\frac{1}{2}\right) \,.$$

   Show that L is unbounded.

4. Show that X is a Banach space if and only if whenever x_n is a sequence such that $\sum_{n=1}^{\infty} \|x_n\| < \infty$, then $\sum_{n=1}^{\infty} x_n$ converges.

5. Suppose that $\tau : [0,1] \to [0,1]$ is a continuous function. Suppose that $A : C[0,1] \to C[0,1]$ is defined by $A[f](x) = f(\tau(x))$ where $C[0,1]$ is equipped with the standard sup norm. Then show that A is bounded with $\|A\| = 1$. Give necessary and sufficient conditions on τ such that a) A is injective and b) A is surjective.

6. Suppose $X = \mathbb{R}^n$. Find sharp constants c and C, such that

$$c\|f\|_\infty \leq \|f\|_1 \leq C\|f\|_\infty \,,$$

   and find the vectors for which the optimal constant is achieved. For a harder version of the problem, replace the ∞ norm by the p norm.