Problem set 7

Due date: Apr 23

May 2, 2018

1. Suppose that R and L are the standard right shift and left shift operators on $\ell^2(\mathbb{N})$. Show that

$$\sigma_p(R) = \emptyset\,, \quad \sigma_c(R) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}\,, \quad \sigma_r(R) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}\,,$$

$$\sigma_p(L) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}\,, \quad \sigma_c(L) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}\,, \quad \sigma_r(L) = \emptyset\,.$$

Solution:
We first observe that $\|R\| = \|L\| = 1$, so that λ with $|\lambda| > 1$ is in the resolvent sets of both L and R. Point spectra $(\sigma_p(L), \sigma_p(R))$: We first show that $R - \lambda I$ is injective for all $\lambda \in \mathbb{C}$. $(R - \lambda)x = 0$ implies $x_{i-1} - \lambda x_i = 0$ for $i = 2, 3, \ldots$ and $-\lambda x_1 = 0$. If $\lambda \neq 0$, then $x_1 = 0$ and an inductive argument shows that $x_i = 0$ for all i. If $\lambda = 0$, then $x_i = 0$ for all i. Thus, $R - \lambda I$ is always injective and hence $\sigma_p(R) = 0$.

We now compute the point spectrum for L. If $|\lambda| < 1$, then $x = (1, \lambda, \lambda^2 \ldots) \in \ell^2(\mathbb{N})$ is an eigenvector for L with eigenvalue λ. If $|\lambda| = 1$, Suppose that $(L - \lambda I)x = 0$, then we have that $x_{i+1} - \lambda x_i = 0$ for all i, i.e. $x_i = \lambda^i x_1$. Since $x \in \ell^2(\mathbb{N})$ and $|\lambda| = 1$, we conclude that $x_1 = 0$ and hence $x_i = 0$ for all i. Thus, $\sigma_p(L) = \{\lambda : |\lambda| < 1\}$.

Since the spectrum is closed and for both L, R, we conclude that $\sigma(L) = \{\lambda : |\lambda| \leq 1\}$. For L and $|\lambda| = 1$, we have shown that if $\lambda \in \sigma_r(L)$, then $\overline{\lambda} \in \sigma_p(L^*) = \sigma_p(R)$. Since the point spectrum of R is empty, we conclude that $\sigma_r(L) = \emptyset$. Thus, $\sigma_c(L) = \{\lambda : |\lambda| = 1\}$.

Now for R, we first show that $|\lambda| < 1 \subset \sigma_r(R)$. This follows from the fact that for any $|\lambda| < 1$, the element $(1, \lambda, \lambda^2, \ldots) \in \ell^2(\mathbb{N}) \perp \mathrm{Ran}(R - \lambda I)$. Again, since the spectrum is closed, we conclude that $\sigma(R) = \{\lambda : |\lambda| \leq 1\}$. If any λ with $|\lambda| = 1 \in \sigma_r(R)$, then we would have that $\overline{\lambda} \in \sigma_p(R^*) = \sigma_p(L)$. However, since $\sigma_p(L) \cap \{\lambda : |\lambda| = 1\} = \emptyset$, we conclude that $\{\lambda : |\lambda| = 1\} \in \sigma_c(R)$.

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2. Suppose that e_i are the standard coordinate vectors in $\ell^2(\mathbb{N})$ and T is the operator defined by

$$Te_i = \lambda_i e_i\,,$$

where λ_i are distinct. Show that $\sigma_r(T) = \emptyset$ and that

$$\lambda \in \overline{\{\lambda_i\}} \setminus \{\lambda_i\} \in \sigma_c(T)$$

Solution: Clearly, $\lambda_i \in \sigma_p(T)$. If $\lambda \neq \lambda_i$ then $(T - \lambda)$ is injective, since $(T - \lambda)x = 0$ implies $(\lambda_i - \lambda)x_i = 0$, and since $\lambda \neq \lambda_i$, we conclude that $x_i = 0$ for all i and hence $x = 0$. So $\sigma_p(T) = \{\lambda_i\}_{i=1}^n$. Next, if $\lambda \neq \lambda_i$, then the range of $(T - \lambda I)$ is dense since all finite linear combinations are in the range of the operator. If $y = (y_1, y_2, \ldots, y_n, 0, 0, \ldots)$, then $(T - \lambda I)x = y$, where $x_i = y_i/(\lambda_i - \lambda)$ for $i = 1, 2, \ldots n$, and 0 otherwise. Thus, $T - \lambda I$ has dense range for all $\lambda \neq \lambda_i$. Thus, the residual spectrum is empty.

The second part follows from the fact that the spectrum is closed.

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3. Construct a compact operator T such that 0 is in a) the point spectrum, b) the continuous spectrum, and c) the residual spectrum.
   Solution: a) $0 \in \sigma_p(T)$. $T = 0$ is a nice compact operator which has 0 in the point spectrum.
   b) $0 \in \sigma_c(T)$. Consider the operator defined by $Te_n = \frac{1}{n}e_n$. We have shown earlier that T is compact. By exercise 2 in this pset, we conclude that $0 \in \sigma_c(T)$.
   c) $0 \in \sigma_r(T)$. Let $Te_n = e_{n+1}/(n+1)$. We have again shown earlier that T is compact. Clearly, T is injective, but $R(T) \perp e_1$. So $0 \in \sigma_r(T)$.

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4. Suppose that $\lambda \in \mathbb{R}$. Show that if $\lambda \in \sigma_p(T)$ and $\lambda \notin \sigma_p(T^*)$, then $\lambda \in \sigma_r(T^*)$.
   Solution: Since $\lambda \in \sigma_p(T)$, there exists $x_0 \neq 0$, such that $(T - \lambda I)x_0 = 0$. Since $\lambda \notin \sigma_p(T^*)$, $T^* - \lambda I$ is injective. Since $\lambda \in \mathbb{R}$, $(T - \lambda I)^* = T^* - \lambda I$. The result then follows from $R(T^* - \lambda I) = \mathcal{N}(T - \lambda I)^\perp$.
