Problem set 6

Due date: Apr 16

April 17, 2018

1. Suppose that $X$ is a Banach space with the norm topology. Show that the Ball $X$ is norm closed if and only if it is weakly closed.

**Solution:** This became a rather silly question. The question was supposed to be to show that a convex set is weakly closed if and only if it is norm closed. A simple application of Hahn Banach shows that $\text{Ball} X$ is weakly closed, and it is clearly norm closed.

2. Sequential non-compactness of $\ell^\infty(\mathbb{N})^*$. Let $b_n : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ be the functions defined by

$$b_n(x) = x_n$$

where $x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)$. Show that $b_n$ does not have any subsequence that converges pointwise, where pointwise convergence in this setup means that for every fixed $x \in \ell^\infty(\mathbb{N})$, $b_n(x)$ is Cauchy.

**Solution:**

The proof is by contradiction. Let us assume that $\{b_{n_k}(x)\}_{k=1}^\infty$ and $n_k < n_{k+1}$ is a subsequence that converges pointwise. The claim is that for the following $x$, the sequence $b_{n_k}(x)$ does not converge.

$$x = \sum_{k=1}^\infty (k \text{ mod } 2) 2^{-n_k}$$

Then $b_{n_k}(x)$ is an alternating sequence of 0’s and 1’s and hence $\{b_{n_k}(x)\}_{k=1}^\infty$ does not converge.

3. Suppose that $X$ is a reflexive Banach space, $M$ is a closed linear subspace of $X$ and that $x_0 \in X \setminus M$, then there is a point $y_0 \in M$ such that $\|x_0 - y_0\| = \text{dist}(x_0, M)$. In order to prove this result show that, the norm on $X$ is lower semicontinuous for the weak topology, and the norm of $X^*$ is lower semicontinuous for the weak-*$*$ topology. Further show that a lower semicontinuous function on a compact set must achieve its minimum. Note that a function $f : X \to \mathbb{R}$ is lower semicontinuous at $x_0$ in a topological space if there exists an open neighborhood $U$ of $x_0$ such that

$$f(x) \geq f(x_0) - \varepsilon \quad \forall x \in U$$
**Solution:** First, we show that the norm is weakly semicontinuous. We wish to find an open neighborhood of $x_0$ such that
\[
\|x\| \geq \|x_0\| - \varepsilon, \quad \forall x \in U.
\]
By Hahn-Banach, there exists $x_0^* \in X^*$ such that $x_0^*(x_0) = \|x_0\|$ and $\|x_0^*\| = 1$. Consider the open neighborhood of $x_0$ defined by
\[
U := \{x : |x_0^*(x - x_0)| < \varepsilon\}.
\]
For all $x \in U$, we have
\[
\varepsilon > |x_0^*(x) - x_0^*(x_0)| \geq \|x_0^*(x_0)\| - \|x_0^*(x)\| = \varepsilon > \|x_0\| - \|x_0^*(x)\|.
\]
\[
\therefore \|x_0\| - \varepsilon < \|x_0^*(x)\| \leq \|x_0^*\| \|x\| = \|x\|.
\]
A similar result holds for the norm of $X^*$ in the weak-* topology.

Now, we show that every semi lower continuous function achieves its minimum on a compact set. Suppose not. Suppose $f$ is semi-lower continuous and $K$ is a compact set. Suppose that $f_0 = \inf_{x \in K} f(x)$. Then for each $x$ in $K$, there exists an $\varepsilon > 0$ such that $f(x) - \varepsilon > f_0$. Since $f$ is lower semi-continuous, there exists an open neighborhood of $x$, denoted by $V(x)$ such that $a < f(y)$ for all $y \in V(x)$. Then $K \subset \bigcup_{x \in K} V(x)$ is an open cover, since $K$ is compact there exists a finite cover, i.e. $K \subset \bigcup_{j=1}^n V(x_j)$. Then it is easy to show that $a < a' = \min_{k=1}^n f(x_k) - \varepsilon_k$ is such that $a' < f(x)$ for all $x \in K$ which contradicts that $a$ is the infimum.

Given both of these pieces of information, we now proceed to the main theorem. Consider $B_{2r}(x_0)$, where $r = \text{dis}(x_0, M)$. Then $B_{2r}(x_0)$ is weakly compact, since $X$ is reflexive, and thus $M \cap B_{2r}(x_0)$ is weakly compact, since $M$ is closed. Moreover $M \cap B_{2r}(x_0)$ is non-empty. Since the norm is weakly semi continuous and $M \cap B_{2r}(x_0)$ is a weakly compact set, we conclude that the norm achieves its minimum on this set.

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4. Show that a sequence $f_n \in C[0, 1]$ equipped with the supremum norm converges weakly if and only if it is norm bounded and pointwise convergent. Give an example of a sequence in the space which converges weakly but not in norm.

**Solution:**

a) $C [0, 1]^* = \text{signed finite measures on } [0, 1]$.

Necessity: Suppose $f_n \to f$ weakly. Then $L(f) = f(x)$ is a bounded linear function. Weak convergence implies $L(f_n) = f_n(x) \to L(f) = f(x)$. Hence $f_n \to f$ pointwise. By part 2 of the previous problem $f_n$ is norm bounded.

Sufficient: Suppose the sequence $f_n$ is norm bounded and converges pointwise to $f$. Then $f_n \to f$ weakly. Let $L(f) = \int_0^1 fd\mu$ where $\mu$ is a finite signed measure be a bounded linear functional on $C[0, 1]$. Then by bounded convergence theorem
\[
\lim_{n \uparrow \infty} L(f_n) = \lim_{n \uparrow \infty} \int_0^1 f_n d\mu = \int_0^1 f d\mu = L(f)
\]
Hence $f_n \uparrow f$ weakly

b) Consider the following sequence of functions

$$f_n(x) = \begin{cases} 
0 & x > \frac{1}{n} \\
1 - |2nx - 1| & 0 < x \leq \frac{1}{n}
\end{cases}$$

Then $f_n(x) \in C[0,1]$, $\sup_n \|f_n\|_\infty = 1$ and $f_n \uparrow 0$ pointwise in $[0,1]$. Thus by part a) $f_n \uparrow 0$ weakly. However $\|f_n - 0\| = 1 \ \forall n$ and hence $f_n$ does not converge to 0 in norm