Problem set 6

Due date: Apr 16

April 17, 2018

1. Suppose that X is a Banach space with the norm topology. Show that the Ball X is norm closed if and only if it is weakly closed.
   Solution: This became a rather silly question. The question was supposed to be to show that a convex set is weakly closed if and only if it is norm closed. A simple application of Hahn Banach shows that $BallX$ is weakly closed, and it is clearly norm closed.

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2. Sequential non-compactness of $\ell^\infty(\mathbb{N})^*$. Let $b_n : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ be the functions defined by
   $$b_n(x) = x_n$$
   where $x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)$. Show that b_n does not have any subsequence that converges pointwise, where pointwise convergence in this setup means that for every fixed $x \in \ell^\infty(\mathbb{N})$, $b_n(x)$ is Cauchy.
   Solution:
   The proof is by contradiction. Let us assume that $\{b_{n_k}(x)\}_{k=1}^{\infty}$ and $n_k < n_{k+1}$ is a subseqeunce that converges pointwise. The claim is that for the following x, the sequence $b_{n_k}(x)$ does not converge.
   $$x = \sum_{k=1}^{\infty} (k \bmod 2)\, 2^{-n_k}$$
   Then $b_{n_k}(x)$ is an alternating sequence of 0's and 1's and hence $\{b_{n_k}(x)\}_{k=1}^{\infty}$ does not converge.

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3. Suppose that X is a reflexive Banach space, M is a closed linear subspace of X and that $x_0 \in X \setminus M$, then there is a point $y_0 \in M$ such that $\|x_0 - y_0\| = \operatorname{dist}(x_0, M)$. In order to prove this result show that, the norm on X is lower semicontinuous for the weak topology, and the norm of X^* is lower semicontinuous for the weak-$*$ topology. Further show that a lower semicontinuous function on a compact set must achieve its minimum. Note that a function $f : X \to \mathbb{R}$ is lower semicontinuous at x_0 in a topological space if there exists an open neighborhood U of x_0 such that
   $$f(x) \geq f(x_0) - \varepsilon \quad \forall x \in U$$

Solution: First, we show that the norm is weakly semicontinuous. We wish to find an open neighborhood of x_0 such that

$$\|x\| \geq \|x_0\| - \varepsilon\,, \quad \forall x \in U\,.$$

By Hahn-Banach, there exists $x_0^* \in X^*$ such that $x_0^*(x_0) = \|x_0\|$ and $\|x_0^*\| = 1$. Consider the open neighborhood of x_0 defined by

$$U := \{x : |x_0^*(x - x_0)| < \varepsilon\}\,.$$

For all $x \in U$, we have

$$\begin{aligned} \varepsilon > |x_0^*(x) - x_0^*(x_0)| &\geq \|x_0^*(x_0)\| - \|x_0^*(x)\| \\ \varepsilon &> \|x_0\| - \|x_0^*(x)\|\,. \\ \therefore \|x_0\| - \varepsilon < \|x_0^*(x)\| &\leq \|x_0^*\|\|x\| = \|x\|\,. \end{aligned}$$

A similar result holds for the norm of X^* in the weak-$*$ topology.

Now, we show that every semi lower continuous function achieves its minimum on a compact set. Suppose not. Suppose f is semi-lower continuous and K is a compact set. Suppose that $f_0 = \inf_{x \in K} f(x)$. Then for each x in K, there exists an $\varepsilon > 0$ such that $f(x) - \varepsilon > f_0$. Since f is lower semi-continuous, there exists an open neighborhood of x, denoted by $V(x)$ such that $a < f(y)$ for all $y \in V(x)$. Then $K \subset \cup_{x \in K} V(x)$ is an open cover, since K is compact there exists a finite cover, i.e. $K \subset \cup_{j=1}^n V(x_j)$. Then it is easy to show that $a < a' = \min_{k=1}^n f(x_k) - \varepsilon_k$ is such that $a' < f(x)$ for all $x \in K$ which contradicts that a is the infinimum.

Given both of these pieces of information, we now proceed to the main theorem. Consider $B_{2r}(x_0)$, where $r = \mathrm{dis}(x_0, M)$. Then $B_{2r}(x_0)$ is weakly compact, since X is reflexive, and thus $M \cap B_{2r}(x_0)$ is weakly compact, since M is closed. Moreover $M \cap B_{2r}(x_0)$ is non-empty. Since the norm is weakly semi continuous and $M \cap B_{2r}(x_0)$ is a weakly compact set, we conclude that the norm achieves its minimum on this set.

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4. Show that a sequence $f_n \in C[0,1]$ equipped with the supremum norm converges weakly if and only if it is norm bounded and pointwise convergent. Give an example of a sequence in the space which converges weakly but not in norm.
   Solution:

   a) $C\,[0,1]^* =$ signed finite measures on $[0,1]$.

   Necessity: Suppose $f_n \to f$ weakly. Then $L\,(f) = f\,(x)$ is a bounded linear function. Weak convergence implies $L\,(f_n) = f_n\,(x) \to L\,(f) = f\,(x)$. Hence $f_n \to f$ pointwise. By part 2 of the previous problem f_n is norm bounded.

   Sufficienct: Suppose the sequence f_n is norm bounded and converges pointwise to f. Then $f_n \to f$ weakly. Let $L\,(f) = \int_0^1 f d\mu$ where μ is a finite signed measure be a bounded linear functional on $C\,[0,1]$. Then by bounded convergence theorem

$$\lim_{n \uparrow \infty} L\,(f_n) = \lim_{n \uparrow \infty} \int_0^1 f_n d\mu = \int_0^1 f d\mu = L\,(f) \tag{1}$$

Hence $f_n \uparrow f$ weakly

b) Consider the following sequence of functions

$$f_n(x) = \begin{cases} 0 & x > \frac{1}{n} \\ 1 - |2nx - 1| & 0 < x \leq \frac{1}{n} \end{cases}$$

Then $f_n(x) \in C[0,1]$, $\sup_n \|f_n\|_\infty = 1$ and $f_n \uparrow 0$ pointwise in $[0,1]$. Thus by part a) $f_n \uparrow 0$ weakly. However $\|f_n - 0\| = 1 \quad \forall n$ and hence f_n does not converge to 0 in norm

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