1. Suppose that $X$ is a Banach space with the norm topology. Show that the Ball $X$ is norm closed if and only if it is weakly closed.

2. Sequential non-compactness of $\ell^\infty(\mathbb{N})^*$. Let $b_n : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ be the functions defined by

$$b_n(x) = x_n$$

where $x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)$. Show that $b_n$ does not have any subsequence that converges pointwise, where pointwise convergence in this setup means that for every fixed $x \in \ell^\infty(\mathbb{N})$, $b_n(x)$ is Cauchy.

3. Suppose that $X$ is a reflexive Banach space, $M$ is a closed linear subspace of $X$ and that $x_0 \in X \setminus M$, then there is a point $y_0 \in M$ such that $\|x_0 - y_0\| = \text{dist}(x_0, M)$. In order to prove this result show that, the norm on $X$ is lower semicontinuous for the weak topology, and the norm of $X^*$ is lower semicontinuous for the weak-* topology. Further show that a lower semicontinuous function on a compact set must achieve its minimum. Note that a function $f : X \to \mathbb{R}$ is lower semicontinuous at $x_0$ in a topological space if there exists an open neighborhood $U$ of $x_0$ such that

$$f(x) \geq f(x_0) - \varepsilon \quad \forall x \in U$$

4. Show that a sequence $f_n \in C[0, 1]$ equipped with the supremum norm converges weakly if and only if it is norm bounded and pointwise convergent. Give an example of a sequence in the space which converges weakly but not in norm.