Problem set 6

Due date: Apr 16

April 5, 2018

1. Suppose that X is a Banach space with the norm topology. Show that the Ball X is norm closed if and only if it is weakly closed

2. Sequential non-compactness of $\ell^\infty(\mathbb{N})^*$. Let $b_n : \ell^\infty(\mathbb{N}) \to \mathbb{R}$ be the functions defined by
$$b_n(x) = x_n$$
where $x = (x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)$. Show that b_n does not have any subsequence that converges pointwise, where pointwise convergence in this setup means that for every fixed $x \in \ell^\infty(\mathbb{N})$, $b_n(x)$ is Cauchy.

3. Suppose that X is a reflexive Banach space, M is a closed linear subspace of X and that $x_0 \in X \setminus M$, then there is a point $y_0 \in M$ such that $\|x_0 - y_0\| = \mathrm{dist}(x_0, M)$. In order to prove this result show that, the norm on X is lower semicontinuous for the weak topology, and the norm of X^* is lower semicontinuous for the weak-$*$ topology. Further show that a lower semicontinuous function on a compact set must achieve its minimum. Note that a function $f : X \to \mathbb{R}$ is lower semicontinuous at x_0 in a topological space if there exists an open neighborhood U of x_0 such that
$$f(x) \geq f(x_0) - \varepsilon \quad \forall x \in U$$

4. Show that a sequence $f_n \in C[0,1]$ equipped with the supremum norm converges weakly if and only if it is norm bounded and pointwise convergent. Give an example of a sequence in the space which converges weakly but not in norm.