Problem set 5

Due date: Apr 9

April 17, 2018

If needed you may assume that the topology on vector spaces is generated by a family of seminorms if that helps.

1. Suppose that $\mathcal{X}$ is a topological vector space (TVS), and let $\mathcal{U}$ be the collection of all open sets containing the origin. Prove the following.
   a) If $U \in \mathcal{U}$, then there is a $V \in \mathcal{U}$ such that $V + V \subset U$.
   b) If $U \in \mathcal{U}$, then there is a V in $\mathcal{U}$ such that $V \subset U$ and $\alpha V \subset V$ for all $|\alpha| \leq 1$.
   Solution: Since $0 + 0 = 0$ and addition is continuous, there exist neighborhoods V_1, V_2 of zero such that $V_1 + V_2 \subset U$. Set $V = V_1 \cap V_2$, then clearly $V + V \subset V_1 + V_2 \subset W$.

   Since $0 \times 0 = 0$ and multiplication is continuous, there exist neighborhoods $(-\varepsilon, \varepsilon)$ and V_1 such that $\alpha V_1 \subset U$ for all $\alpha \in (-\varepsilon, \varepsilon)$. Define $V = \varepsilon V$, then clearly $\alpha V \subset U$ for all $\alpha \in (-1, 1)$.

---

2. Suppose that $\mathcal{X}$ is a vector space whose topology is defined by a family of seminorms $\mathcal{P}$, i.e., every open neighborhood of x_0 is of the form

$$U_{x_0} = \cap_{j=1}^{n} \{x \in \mathcal{X} : p_j(x - x_0) < \varepsilon_j\} \,,$$

   where the seminorms further satisfy

$$\cap_{p \in \mathcal{P}} \{x : p(x) = 0\} = (0) \,.$$

   Then show that $\mathcal{X}$ is a topological vector space with this topology.
   Solution: All we need to show that if $f(x, y) = x + y$ and $f(\alpha, x) = \alpha x$ are continuous functions. Continuity of $x + y$. The generating sets in the product topology centered at x_0, y_0 are of the form

$$U_{x_0, y_0, \varepsilon} = \{(x, y) \in \mathcal{X} \times \mathcal{X} : |p_j(x - x_0)| < \varepsilon \quad \text{and} \quad |p_j(y - y_0)| < \varepsilon\} \,.$$

   Suppose that A is an open set in X given by

$$A =: \{x : |p_j(x - x_0| < \varepsilon\} \,.$$

   Let $B = f^{-1}(A)$ and suppose that $(x_1, y_1) \in B$. We need to find an open neighborhood of B such that its image under $x + y$ is completely contained in A. Since $(x_1, y_1) \in B$, $x_1 + y_1 \in A$ and satisfies

$$|p_j(x_1 + y_1 - x_0)| = c_0 < \varepsilon \,.$$

Suppose that ε' is such that $c_0 + \varepsilon' < \varepsilon$. Consider the open set in $X \times X$ given by $U_{x_1,y_1,\varepsilon'/2}$, i.e.

$$U_{x_1,y_1,\varepsilon'/2} = \{(x,y) \in \mathcal{X} \times \mathcal{X} : |p_j(x-x_1)| < \varepsilon'/2 \quad \text{and} \quad |p_j(y-y_1)| < \varepsilon'/2\}\,.$$

Clearly, $(x_1, y_1) \in U_{x_1,y_1,\varepsilon'/2}$. Moreover, for all $(x,y) \in U_{x_1,y_1,\varepsilon'/2}$,

$$\begin{aligned} |p_j(f(x,y) - x_0)| &= |p_j(x + y - x_0)| \\ &\leq |p_j(x - x_1| + |p_j(y - y_1)| + |p_j(x_1 + y_1 - x_0)| \\ &\leq \varepsilon'/2 + \varepsilon'/2 + c_0 < \varepsilon\,. \end{aligned}$$

Thus for all $(x,y) \in U_{x_1,y_1,\varepsilon'/2}$, $f(x,y) \in A$, and thus $U \subset f^{-1}(A)$. Thus B is an open neighborhood of $X \times X$ and addition is a continuous function.

A similar proof holds for multiplication.

---

3. Let $\mathcal{X}$ be a TVS. Show: a) if $x_0 \in \mathcal{X}$, then the mapping $x \to x + x_0$ is a homeomorphism, i.e., a continuous function and continuous inverse; b) if $\alpha \in \mathcal{F}$, and $\alpha \neq .0$, the map $x \to \alpha x$ is a homeomorphism.

Solution: Let $f(x) = x + x_0$. Let τ be the topology on X generated by a collection of semi-norms. Then let A be an open set of the form

$$A = \{x : |p_j(x - x_1)| < \varepsilon\}\,.$$

and let $B = f^{-1}(A)$. Suppose that $x_2 \in B$, then

$$|p_j(x_2 + x_0 - x_1)| = c_0 < \varepsilon\,.$$

Let ε' such that $c_0 + \varepsilon' < \varepsilon$. Then consider the open set V centered containing x_2 given by

$$V = \{x : |p_j(x - x_2)| < \varepsilon'\}\,,$$

Then

$$\begin{aligned} f(V) &= \{x + x_0 : |p_j(x - x_2| < \varepsilon'\} \\ &= \{x : |p_j(x - x_0 - x_2)| < \varepsilon'\} \\ &= \{x : |p_j(x - x_1 + x_1 - x_0 - x_2)| < \varepsilon'\} \end{aligned}$$

We note that

$$\begin{aligned} |p_j(x - x_0 - x_2)| < \varepsilon' \implies |p_j(x - x_1)| &= |p_j(x - x_0 - x_2 + x_0 + x_2 - x_1)| \\ &\leq |p_j(x - x_0 - x_2)| + |p_j(x_0 + x_2 - x_1)| \\ &\leq c_0 + \varepsilon' < \varepsilon\,. \end{aligned}$$

Thus, $f(V) \subset A$ or $V \subset f^{-1}(A)$. Thus, B is open since every point in B contains an open neighborhood. This shows the continuity of f. Clearly, f has an inverse defined by $f^{-1}(x) = x - x_0$ which is continuous by the same argument.

The proof for $x \to \alpha x$ being a homeomorphism follows in a similar manner.

---

4. Show that the weak topology is the smallest topology on $\mathcal{X}$ such that each $x^* \in \mathcal{X}^*$ is continuous.
Solution: First let us show that x^* are continuous in the weak topology. Consider any open set $(a,b) \in \mathbb{R}$. Let $U = (x^*)^{-1}(a,b)$. Suppose that $x_0 \in U$. Let $x^*(x_0) = \alpha_0 \in (a,b)$. Let $\varepsilon > 0$ such that $(\alpha_0-\varepsilon, \alpha_0+\varepsilon) \in (a,b)$. Then $U_{x_0,\varepsilon} = \{x : |x^*(x-x_0)| < \varepsilon\}$ is an open set contained in U, since for all $x \in U$, $x^*(x_0)-\varepsilon \le x^*(x) \le x^*(x_0)+\varepsilon$. Thus, every $x_0 \in U$, contains a weakly open neighborhood around x_0. Thus U is weakly open and hence x^* is continuous in the weak topology.

Now suppose that τ is a topology in which all x^* are continuous. Fix an $x^* \in x$, then Then
$$U_{x_0,\varepsilon} := \{x : |x^*(x-x_0)| < \varepsilon\} \in \tau\,,$$
since it is the inverse image of an open set of a continuous function. This is precisely the generator sets for the weak topology.

Remark 1. *The proof of problem 5 is similar.*

---

5. Show that the weak$-*$ topology is the smallest topology on $\mathcal{X}^*$ such that each $x \in \mathcal{X}$, $x^* \to x^*(x)$ is continuous.

---

6. If $\mathcal{H}$ is a Hilbert space and $\{h_n\}$ is a sequence in $\mathcal{H}$ such that $h_n \to h$ weakly, i.e. $(h_n, f) \to (h, f)$ as $n \to \infty$ for each $f \in \mathcal{H}$. Suppose further that $\|h_n\| \to \|h\|$, then show that $\|h_n - h\| \to 0$.
Solution:
$$\|h_n - h\|^2 = \|h_n\|^2 + \|h^2\| - 2(h_n, h) \to 0\,,$$
since $(h_n, h) \to (h, h)$ and $\|h_n\|^2 \to \|h\|^2$.

---

7. Suppose that $\mathcal{X}$ is an infinite-dimensional normed space. If $\mathcal{S} = \{x \in \mathcal{X} : \|x\| = 1\}$, then the weak closure of S is $\{x : \|x\| \le 1\}$.
Solution: Let $\overline{\mathcal{S}}$ denote the weak closure of $\mathcal{S}$. First suppose that x_0 is such that $\|x_0\| > 1$. Then by the Hahn-Banach there exists an ℓ such that
$$|\ell(x-x_0)| > \varepsilon\,, \quad \text{for all} x \,\text{such that} \|x\| \le 1\,.$$
Thus, there exists an open neighborhood U of x_0 given by
$$U := \{x : |\ell(x-x_0)| < \varepsilon\}\,,$$
such that $U \cap \mathcal{S} = \emptyset$. Hence, if $\|x_0\| > 1$, then x_0 is not in the weak closure of $\mathcal{S}$.

This shows that $\overline{\mathcal{S}} \subset \{x : \|x\| \le 1\}$. Now suppose that x_0 is such that $\|x_0\| < 1$, and suppose that $x_0 \in \overline{\mathcal{S}}^c$. Since $\overline{\mathcal{S}}^c$ is weakly open, there must exist an open neighborhood of x_0, U, such that $U \subset \overline{\mathcal{S}}^c$. However, consider any open neighborhood of x_0 which takes the form
$$U =: \cap_{j=1}^n x : |x_j^*(x-x_0)| < \varepsilon\,.$$

Claim, there exists $y \in \mathcal{S}$ such that $y \in U$. Since X is infinite dimensional, suppose that $x_1, x_2, \dots x_{n+1}$ are linearly independent elements of X. Now, consider the collection of vectors $\sum_{j=1}^{n+1} c_j x_j$, and consider the linear system formed by

$$\boldsymbol{x}_j^{ast}(\sum_{k=1}^{n+1} c_k x_k) = 0\,,$$

This is a linear mapping from $\mathbb{R}^{n+1} \to \mathbb{R}^n$ for the coefficients c_j. By the rank-nullity theorem, there exists a vector $\{d_j\}_{j=1}^n$ in the null space of the mapping, i.e.

$$x_j^*(\sum_{k}^{n+1} d_k x_k) = 0\,.$$

Thus, $y = \sum_{j=1}^{n+1} d_j x_j \in U$. By the same argument $y/\|y\| \in U$ is the element in $\mathcal{S}$ we are looking for. Thus, every open neighborhood of x_0 intersects with $\mathcal{S}$ and hence x_0 cannot be in $\overline{\mathcal{S}}^c$.

---

8. In an infinite dimensional vector space, show that a bounded set cannot be open in the weak topology.
Solution: We shall show the contrapositive, that every open set in the weak topology is unbounded. Consider a point x, the neighborhood basis in the weak topology which contain x are sets of the form $V_{f_1,f_2\dots f_n,\epsilon}(x) = \{y \in X,\ |f_i(y) - f_i(x)| < \epsilon \quad \forall i = 1,2,\dots n\}$ where $f_i \in X^*$ and $n \in \mathbb{N}$. So without loss of generality, we can assume $x = 0$ since

$$V_{f_!,f_2\dots\dots f_n,\epsilon}(x) = \{y + x \in X;\ |f_i(y)| < \epsilon\} = \{y + x;\ y \in V_{f_1,f_2\dots f_n,\epsilon}(0)\} = V_{f_1,f_2\dots f_n,\epsilon}(0) + x$$

Since we are in an infinite dimensional vector space, then $\exists x_i$, such that, $\|x_i\| = 1$ and $d(x_i, \operatorname{span}(x_1, x_2 \dots x_{i-1})) \geq \frac{1}{2}, \quad \forall i = 1,2,\dots n+1$. Let $x \in \operatorname{span}\{x_1, x_2 \dots x_{n+1}\}$. Then $x = \sum_{j=1}^{n+1} c_j x_j$.

$$\begin{bmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_{n-1}(x) \\ f_n(x) \end{bmatrix} = \begin{bmatrix} f_1(x_1) & f_1(x_2) & \dots & f_1(x_n) & f_1(x_{n+1}) \\ f_2(x_1) & f_2(x_2) & \dots & f_2(x_n) & f_2(x_{n+1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ f_n(x_1) & f_n(x_2) & \dots & f_n(x_n) & f_n(x_{n+1}) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \\ c_{n+1} \end{bmatrix}$$

Thus the mapping of $[c_1, c_2 \dots c_{n+1}] \to [f_1(x), f_2(x) \dots f_n(x)]$ is a linear map from $\mathbf{R}^{n+1} \to \mathbf{R}^n$. Hence by the rank nullity theorem, $\exists c \neq 0$ such that $f_i(x) = 0$, $\forall i = 1,2,\dots n$. That is $f_i\left(\sum_{j=1}^{n+1} c_j x_j\right) = 0$ for all $i = 1,2,\dots n$. The claim is that $\sum_{j=1}^{n+1} c_j x_j \neq 0$. Suppose not. Then let $j_m = \max_{j=1,2,\dots n+1}\{c_j \neq 0\}$. $j_m > 0$ since $c \neq 0$. If $j_m > 1$, then $x_{j_m} = \sum_{j=1}^{j_m-1} \overline{c}_j x_j$ contradicting the fact that $d(x_{j_m}, \operatorname{span}(x_1, x_2 \dots x_{j_m-1})) \geq$

$\frac{1}{2}$. If $j_m = 1$, then $c_1 x_1 = 0$, contradicting the fact that $\|x_1\| = 1$. Hence $x = \sum_{j=1}^{n+1} c_j x_j \in \cap_{i=1}^{n} \mathrm{Ker}\,(f_i)$ and $K = \cap_{i=1}^{n} \mathrm{Ker}\,\{f_i\}$ is a linear subspace of $V_{f_1, f_2 \cdots f_n, \epsilon}\,(0)$. Hence if $x \in K$, $\alpha x \in K$ for all $\alpha \in \mathcal{F}$. Since we have shown that K is non trivial, that is there $\exists x \neq 0$ and $x \in K$, we conclude that $V_{f_1, f_2, \cdots f_n, \epsilon}\,(0)$ is unbounded for any $n \in \mathbb{N}$ and for all $\epsilon > 0$. Every open set (open in the weak topology) which contains 0, is the union of such sets. Hence every open set containing 0, must be unbounded which implies that every weakly open set containing a point is unbounded. Hence a bounded set cannot be weakly open.
