Problem set 4

Due date: Mar 26

April 17, 2018

1. Suppose that X, Y are Banach spaces. If A is a bounded linear operator and $\text{ran}(A)$ is second category, then show that $\text{ran}(A)$ is closed.
Solution:
The proof is exactly the same as that of open mapping theorem.

$$\text{Ran}(A) = \cup_{n=1}^{\infty} nA(B_1(0)) \, .$$

Since $\text{Ran}(A)$ is second category, we conclude that $\overline{A(B_1(0))}$ has a non-empty interior (Since $nA(B_1(0))$ is homeomorphic to $A(B_1(0))$. Following the proof of the open mapping theorem, we conclude that $B_\varepsilon(0) \subset A(B_1(0))$ for some $\varepsilon > 0$. From which we conclude that $\text{Ran}(A) = Y$ and hence $\text{Ran}(A)$ is closed.

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2. Suppose that $C^1[0,1]$, the space of continuously differentiable functions on $[0,1]$ and $C[0,1]$ the space of continuous functions are both equipped with the supremum norm. Suppose that $A : C^1[0,1] \to C[0,1]$ be defined by $Af = f'$, then show that A is unbounded.
Solution:
Consider the sequence $f_n(x) = e^{inx}$, then $\sup_{[0,1]} |f_n| = 1$, however,

$$\sup_{[0,1]} |Af_n| = \sup_{[0,1]} |f_n'| = \sup_{[0,1]} |ne^{inx}| = n \, .$$

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3. Suppose X, Y are Banach spaces. Show that there is a constant $c > 0$ such that $\|Ax\| \geq c\|x\|$ if and only if $\mathcal{N}(A) = \{0\}$ and $\text{ran}(A)$ is closed.
Solution:
Suppose $\|Ax\| \geq c\|x\|$ for some $c > 0$. Then

$$Ax = 0 \implies c\|x\| \leq 0 \implies x = 0 \, .$$

Thus, $\mathcal{N}(A) = \{0\}$. Now suppose $Af_n \to g$ for some $g \in Y$. Furthermore, the sequence Af_n is Cauchy.

$$c\|f_n - f_m\| \leq \|Af_n - Af_m\| \leq \varepsilon \quad \text{for n,m sufficiently large} \, .$$

Thus, the sequence f_n is Cauchy. Since X is a Banach space, $f_n \to h$ for some $h \in X$. By the continuity of A, we then conclude that

$$g = \lim_n Af_n = A \lim_n f_n = Ah$$

, from which we conclude that g is in the range of A and hence the range of A is closed.

Now suppose $\mathcal{N}(A) = \{0\}$ and that the range of A is closed. Since closed subspaces of Banach spaces are themselves, Banach spaces, we conclude that $\operatorname{ran}(A)$ is a Banach space. Then $A : X \to \operatorname{ran}(A)$ is a surjective operator between Banach spaces and hence by the inverse mapping theorem is an invertible operator. Thus, for all $f \in X$,

$$\|A^{-1}(Af)\| \leq \|A^{-1}\|\|Af\| \implies \frac{1}{\|A^{-1}\|}\|f\| \leq \|Af\| \, .$$

where $0 < \|A^{-1}\| < \infty$ which completes the proof.

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4. If $1 < p < \infty$ and $\{x_n\} \in \ell^p$, then $\sum_{j=1}^{\infty} x_n^j y^j \to 0$ for every $y \in \ell^q$, $1/p + 1/q = 1$, if and only if $\sup_n \|x_n\|_{\ell^p} < \infty$ and $x_n^j \to 0$ as $n \to \infty$ for all j.
Solution:
Suppose that $\sum_{j=1}^{\infty} x_n^j y_j \to 0$ for every $y \in \ell^q$. Then consider the sequence of operators $\ell_n \in (\ell^q)^*$ defined via

$$\ell_n(y) = (x_n, y) = \sum_{j=1}^{\infty} x_n^j y_j \, .$$

Since each $x_n \in \ell^p$, from Holder's inequality it follows that each of ℓ_n are indeed bounded linear operators. Moreover,

$$\|\ell_n\| = \|x_n\|_{\ell^p} \, .$$

Since ℓ_n are pointwise convergent, and hence pointwise bounded, we conclude from the uniform boundedness principle that $\sup_n \|x_n\|_{\ell^p} < \infty$. To show that $x_n^j \to 0$ as $n \to \infty$, set $y = e_k$, where e_k are the standard coordinate vectors. Then

$$\sum_{j=1}^{\infty} x_n^j y^j = x_n^k \to 0 \quad \text{as} \quad n \to \infty \, .$$

Now suppose that $\sup_n \|x_n\| = M < \infty$ and that $x_n^j \to 0$ as $n \to \infty$ for each j. For a fixed y, choose N large enough so that

$$\left(\sum_{j=N+1}^{\infty} |y_j|^q \right)^{1/q} \leq \varepsilon / M$$

. Then,

$$
\begin{aligned}
|\sum_{j=1}^{\infty} x_n^j y^j| &= |\sum_{j=1}^{N} x_n^j y^j + \sum_{N+1}^{\infty} x_n^j y^j| \\
&\leq |\sum_{j=1}^{N} x_n^j y_j| + \sum_{N+1}^{\infty} |x_n^j y^j| \\
&\leq |\sum_{j=1}^{N} x_n^j y_j| + \|x_n\|_{\ell^p} \left(\sum_{j=N+1}^{\infty} |y_j|^q \right)^{1/q} \\
&\leq |\sum_{j=1}^{N} x_n^j y_j| + \varepsilon \\
&\leq 2\varepsilon \,,
\end{aligned}
$$

where the first sum goes to zero as $n \to \infty$ since for each $n \in \mathbb{N}$ and $j = 1, 2, \ldots N$, $x_n^j \to 0$ and y^j are bounded.

Remark 1. *The proofs of problems 5 and 6 are similar.*

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5. If $\{x_n\} \in \ell^1$, then $\sum_{j=1}^{\infty} x_n^j y^j \to 0$ for every $y \in c_0$, if and only if $\sup_n \|x_n\|_{\ell^1} < \infty$ and $x_n^j \to 0$ as $n \to \infty$ for all j.

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6. Suppose $\mathcal{H}$ is a separable Hilbert space with basis e_i. Show that a sequence $h_n \in \mathcal{H}$ satifies $(h_n, h) \to 0$ for every $h \in \mathcal{H}$ if and only if $\sup_n \|h_n\| < \infty$ and $(h_n, e_j) \to 0$ as $n \to \infty$ for each j.

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7. Suppose that X and Y are Banach spaces, and let $A_n : X \to Y$ are pointwise convergent with limit $A : X \to Y$, i.e.

$$A_n f \to Af \quad \forall f \in X \,.$$

Then the convergence is uniform on compact subsets U of X, that is,

$$\sup_{f \in U} \|A_n \phi - A\phi\| \to 0 \quad n \to \infty \,.$$

Solution:
From Banach Steinhaus, it follows that $\sup_n \|A_n\| \leq M$ and $\|A\| \leq M$. For $\varepsilon > 0$, consider the open balls $B(\phi, r) = \{\psi \in X : \|\psi - \phi\| < r\}$, with center $\phi \in X$ and radius $r = \varepsilon/(3M)$. Then,

$$U \subset \cup_{\phi \in U} B(\phi, r) \,,$$

forms an open covering of U. Since U is compact, there exists a finite cover, i.e.

$$U \subset \cup_{j=1}^{m} B(\phi_j, r) \,.$$

Since the operators A_n are pointwise convergent to A, there exists $N(\varepsilon)$ such that $n > N(\varepsilon)$ implies

$$\|A_n\phi_j - A\phi\| \leq \frac{\varepsilon}{3} \quad \text{for all } j = 1, 2, \dots m\,.$$

Then for any $\phi \in U$, ϕ is contained in some ball $B(\phi_j, r)$, i.e., $\|\phi_j - \phi\| < r$. For all $n > N(\varepsilon)$, we then have that

$$\begin{aligned}\|A_n\phi - A\phi\| &\leq \|A_n\phi - A_n\phi_j\| + \|(A_n - A)\phi_j\| + \|A(\phi_j - \phi)\| \\ &\leq \|A_n\|\|\phi_j - \phi\| + \|A\|\|\phi_j - \phi\| + \|(A - A_n)\phi_j\| \\ &\leq M \cdot \frac{\varepsilon}{3M} + M \cdot \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} = \varepsilon\,.\end{aligned}$$

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8. A family of bounded operators $\mathcal{A} = \{A : X \to Y\}$ of linear operators, where X, Y are banach spaces are called collectively compact, if for each bounded set $U \subset X$, the image set $\mathcal{A}(U) = \{Af : f \in U, A \in \mathcal{A}\}$ is relatively compact in Y. Suppose X, Y, and Z are banach spaces. Let $L_n : Y \to Z$ converge pointwise with the limit operator $L : Y \to Z$ (see definition above for pointwise convergence), and let $\mathcal{A}$ be a collection of collectively compact operators. Then

$$\sup_{A \in \mathcal{A}} \|(L_n - L)A\| \to 0 \quad n \to \infty\,.$$

Solution:
Set $U = \{A\phi : \|\phi| \leq 1 \quad A \in \mathcal{A}\}$. Then by assumption, U is relatively compact. By the previous result, the convergence $L_n\phi \to L\phi$ is uniform on U. Thus, for every $\varepsilon > 0$, there exists an integer $N(\varepsilon)$ such that

$$\|(L_n - L)A\phi\| < \varepsilon\,,$$

for all $n \geq N(\varepsilon)$, all ϕ with $\|\phi\| \leq 1$, and all $A \in \mathcal{A}$. This also implies that

$$\|(L_n - L)A\| \leq \varepsilon\,,$$

for all $n > N(\varepsilon)$ and all $A \in \mathcal{A}$.

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9. Suppose that A_n is a collection of collectively compact operators which converge pointwise to A. Suppose further that A is a compact operator. Assume that $I - A$ is injective. Then for all sufficiently large n, for all n with

$$\|(I - A)^{-1}(A_n - A)A_n\| < 1\,,$$

the operators $I - A_n$ are invertible and uniformly bounded by

$$\|(I - A_n)^{-1}\| \leq \frac{1 + \|(I - A)^{-1}A_n\|}{1 - \|(I - A)^{-1}(A_n - A)A_n\|}\,,$$

For the solutions of the equations

$$\phi - A\phi = f \quad \text{and} \quad \phi_n - A_n\phi_n f \,,$$

there holds the error estimate

$$\|\phi_n - \phi\| \leq \|(I-A)^{-1}\| \frac{\|(A_n - A)f\| + \|(A_n - A)A_n\phi\|}{1 - \|(I-A)^{-1}(A_n - A)A_n\|}$$

Solution:
By the Fredholm alternative, $\|(I-A)^{-1}\| < \infty$. We note that

$$(I + (I-A)^{-1}A_n)(I - A_n) = I - (I-A)^{-1}(A_n - A)A_n = I - S_n \,.$$

From the pervious theorem, $\|(A_n - A)A_n\| \to 0$ as $n \to \infty$. Thus, for sufficiently large n, $\|(I-A)^{-1}(A_n - A)A_n\| = \|S_n\| < 1$, since $(I-A)^{-1}$ is a bounded operator. It then follows from the Neumann series that $I - S_n$ is invertible and that

$$\|I - S_n\| \leq \frac{1}{1 - \|S_n\|} \,.$$

Now, we note that

$$\mathcal{N}(I - A_n) \subset \mathcal{N}((I + (I-A)^{-1}A_n)(I - A_n)) = \mathcal{N}(I - S_n) \subset \{0\} \,.$$

Thus, $I - A_n$ is injective. It then follows from the Fredholm alternative that $I - A_n$ is surjective and has a bounded inverse. Moreover,

$$(I - A_n)^{-1} = (I - S_n)^{-1}(I + (I-A)^{-1}A_n) \,.$$

A simple calculation, then shows that

$$\begin{aligned}
(I - A_n)^{-1} - (I-A)^{-1} &= (I - S_n)^{-1}(I + (I-A)^{-1}A_n - (I - S_n)(I-A)^{-1}) \\
&= (I - S_n)^{-1}((I-A)^{-1}(I - A + A_n - I) + S_n(I-A)^{-1}) \\
&= (I - S_n)^{-1}((I-A)^{-1}(A_n - A) + S_n(I-A)^{-1}) \,,
\end{aligned}$$

from which the result follows.