Problem set 4

Due date: Mar 26

March 25, 2018

1. Suppose that X, Y are Banach spaces. If A is a bounded linear operator and $\text{ran}(A)$ is second category, then show that $\text{ran}(A)$ is closed.

2. Suppose that $C^1[0,1]$, the space of continuously differentiable functions on $[0,1]$ and $C[0,1]$ the space of continuous functions are both equipped with the supremum norm. Suppose that $A : C^1[0,1] \to C[0,1]$ be defined by $Af = f'$, then show that A is unbounded.

3. Suppose X, Y are Banach spaces. Show that there is a constant $c > 0$ such that $\|Ax\| \geq c\|x\|$ if and only if $\mathcal{N}(A) = \{0\}$ and $\text{ran}(A)$ is closed.

4. If $1 < p < \infty$ and $\{x_n\} \in \ell^p$, then $\sum_{j=1}^{\infty} x_n^j y^j \to 0$ for every $y \in \ell^q$, $1/p + 1/q = 1$, if and only if $\sup_n \|x_n\|_{\ell^p} < \infty$ and $x_n^j \to 0$ as $n \to \infty$ for all j.

5. If $\{x_n\} \in \ell^1$, then $\sum_{j=1}^{\infty} x_n^j y^j \to 0$ for every $y \in c_0$, if and only if $\sup_n \|x_n\|_{\ell^1} < \infty$ and $x_n^j \to 0$ as $n \to \infty$ for all j.

6. Suppose $\mathcal{H}$ is a separable Hilbert space with basis e_i. Show that a sequence $h_n \in \mathcal{H}$ satifies $(h_n, h) \to 0$ for every $h \in \mathcal{H}$ if and only if $\sup_n \|h_n\| < \infty$ and $(h_n, e_j) \to 0$ as $n \to \infty$ for each j.

7. Suppose that X and Y are Banach spaces, and let $A_n : X \to Y$ are pointwise convergent with limit $A : X \to Y$, i.e.
$$A_n f \to Af \quad \forall f \in X\,.$$
Then the convergence is uniform on compact subsets U of X, that is,
$$\sup_{f \in U} \|A_n \phi - A\phi\| \to 0 \quad n \to \infty\,.$$

8. A family of bounded operators $\mathcal{A} = \{A : X \to Y\}$ of linear operators, where X, Y are banach spaces are called collectively compact, if for each bounded set $U \subset X$, the image set $\mathcal{A}(U) = \{Af : f \in U, A \in \mathcal{A}\}$ is relatively compact in Y. Suppose X, Y, and Z are banach spaces. Let $L_n : Y \to Z$ converge pointwise with the limit operator $L : Y \to Z$ (see definition above for pointwise convergence), and let $\mathcal{A}$ be a collection of collectively compact operators. Then
$$\sup_{A \in \mathcal{A}} \|(L_n - L)A\| \to 0 \quad n \to \infty\,.$$

9. Suppose that A_n is a collection of collectively compact operators which converge pointwise to A. Suppose further that A is a compact operator. Assume that $I - A$ is injective. Then for all sufficiently large n, for all n with

$$\|(I-A)^{-1}(A_n - A)A_n\| < 1\,,$$

the operators $I - A_n$ are invertible and uniformly bounded by

$$\|(I-A_n)^{-1}\| \leq \frac{1 + \|(I-A)^{-1}A_n\|}{1 - \|(I-A)^{-1}(A_n - A)A_n\|}\,,$$

For the solutions of the equations

$$\phi - A\phi = f \quad \text{and} \quad \phi_n - A_n\phi_n f\,,$$

there holds the error estimate

$$\|\phi_n - \phi\| \leq \|(I-A)^{-1}\| \frac{\|(A_n - A)f\| + \|(A_n - A)A_n\phi\|}{1 - \|(I-A)^{-1}(A_n - A)A_n\|}$$