# Problem set 3

Due date: Mar 5

March 6, 2018

1. Suppose that $\ell^1(\mathbb{N})$ is the space of sequences which are absolutely summable, i.e.

$$\{a_n\} \in \ell^1(\mathbb{N}) \quad \text{if} \quad \sum_{n=1}^{\infty} |a_n| < \infty\,.$$

Suppose that $c_0(\mathbb{N})$ is the space of sequences that converge to 0, i.e.

$$\{a_n\} \in c_0(\mathbb{N}) \quad \text{if} \quad \lim_{n\to\infty} a_n = 0\,.$$

Show that $c_0(\mathbb{N})^* = \ell^1(\mathbb{N})$.
**Solution**: Since for every $a \in \ell^1$, inner product with $a$ is a continuous function on $\ell^\infty$, and $c_0 \subset \ell^\infty$, we conclude that $a \in \ell^1$ is also a continuous function on $\ell^\infty$.

Now suppose $\ell \in (c_0)^*$, and let $e_i$ denote the standard coordinate vectors. Let $a_n = \ell(e_n)$, for each $n$. If $x$ is finitely supported, i.e. $x = \sum_{n=1}^{N} x_n e_n$, where $x_n \in \mathbb{R}$, then by linearity of $\ell$,

$$\ell(x) = \sum_{n=1}^{N} a_n x_n\,.$$

Choose

$$x_k^N . = \begin{cases} \frac{|a_k|}{a_k} & 1 \leq k \leq N \\ 0 & \text{otherwise}\,. \end{cases}$$

Then for each $N$, $\|x_N\|_{\ell^\infty} = 1$, and

$$\ell(x^N) = \sum_{n=1}^{N} |a_n|\,.$$

However, $\ell$ is a bounded linear functional, so that

$$\sum_{n=1}^{N} |a_n| = \ell(x_N) \leq \|\ell\| \|x^N\| = \|\ell\| < \infty\,.$$

Since the result is true for all $N$, we conclude that

$$\sum_{n=1}^{\infty} |a_n| < \infty\,.$$

Thus, every element in the dual can be identified as an inner product with a sequence in $\ell^1$.

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2. Suppose that $\ell^\infty(\mathbb{N})$ is the space of sequences which are bounded, i.e.

$$\{a_n\} \in \ell^\infty(\mathbb{N}) \quad \text{if} \quad \sup_n |a_n| < \infty\,.$$

Show that $\ell^\infty(\mathbb{N})$ is not separable.
**Solution**: Consider the binary representation of all real numbers and list it as a sequence in $\ell^\infty$. Then the $\ell^2$ norm between any two distinct numbers is at least 1. Thus, there exists an uncountable number of elements on the vector space which are linearly independent and hence the space is not separable.

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3. Suppose that $c \subset \ell^\infty(\mathbb{N})$ is the space of sequences that converge, i.e.

$$\{a_n\} \in c \quad \text{if} \quad \lim_{n\to\infty} a_n \quad \text{exists}\,.$$

Show that $c$ is a closed subspace.
**Solution**: Suppose that $a_n = \{a_{n,k}\}_{k=1}^\infty$ and that $a_n \to b = \{b_k\}_{k=1}^\infty$, as $n \to \infty$. From this it follows that $a_{n,k} \to b_k$ as $n \to \infty$ for each $k$. We now wish to show that the sequence $b_k$ is Cauchy. Then, let $\varepsilon > 0$. Suppose that $N$ is such that $\|a_n - b\|_{\ell^\infty} \leq \varepsilon$, for all $n \geq N_1$, i.e. for all $n \geq N$ and all $k$ $|a_{n,k} - b_k| < \varepsilon$. Then,

$$|b_n - b_m| = |b_n - a_{N_1,n} + a_{N_1,m} - a_{N_1,m} - b_m| \leq |b_n - a_{N_1,n}| + |b_m - a_{N_1,m}| + |a_{N_1,n} - a_{N_1,m}| \leq 3\varepsilon$$

The first two terms are less than $\varepsilon$ since, $|a_{n,k} - b_k| \leq \varepsilon$ for all $k$ and the last term is less than $\varepsilon$ since $a_{N_1}$ is a convergent sequence. Thus, $b$ is a Cauchy sequence and converges.

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4. (Optional) Recall that the dual of $C[0,1]$ is the space of Borel measures on the interval $[0,1]$. Construct a bounded linear functional in $(C[0,1])^*$ which does not attain its norm.

5. Let

$$\ell(f) = f(x_0)\,,$$

denote a linear functional in $(C[0,1])^*$ where $0 < x_0 < 1$. Show that $\ell$ is a bounded and find the norm of $\ell$.
**Solution**:

$$|\ell(f)| = |f(x_0)| \leq \sup |f|\,.$$

Thus, $\|\ell\| \leq 1$. In fact $\|\ell\| = 1$. To show that the norm is achieved, consider any function which achieves its maximum absolute value at $x_0$.

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6. Suppose that $X = L^2[-1,1]$. For each scalar $\alpha$, let

$$E_\alpha := \{f \in C[-1,1], f(0) = \alpha\}.$$

Show that

(a) Each $E_\alpha$ is convex and dense in $X$

(b) For $\alpha \neq \beta$, $E_\alpha$, $E_\beta$ are disjoint but there is no continuous functional on $\ell$ on $X$ such that

$$\sup_{f\in E_\alpha} \ell(f) \leq \inf_{f\in E_\beta} \ell(f).$$

Explain why geometric Hahn-Banach could not be employed.

a) Let $f, g \in E_\alpha$, then $\forall t \in [0,1]$, $tf + (1-t)\,g \in C\,[-1,1]$ and $tf\,(0) + (1-t)\,g\,(0) = \alpha$. Thus $tf + (1-t)\,g \in E_\alpha$. Hence $E_\alpha$ is convex.

Density: Let $\epsilon > 0$, $f \in \mathbb{L}^2\,[-1,1]$. Continuous funcitons are dense in $\mathbb{L}^2\,[-1,1]$. Let $h \in C\,[-1,1]$ be such that $|h - f|_{\mathbb{L}^2[-1,1]} < \frac{\epsilon}{2}$.

$$g_\delta\,(x) = \begin{cases} h\,(x) & |x| > \delta \\ \frac{\alpha}{\delta}\,(x+\delta) + h\,(-\delta) & -\delta \leq x \leq 0 \\ -\frac{\alpha}{\delta}\,(x-\delta) + h\,(\delta) & 0 \leq x \leq \delta \end{cases}$$

$g_\delta\,(x)$ is a continuous function such that $g_\delta\,(0) = \alpha$. Thus $g_\delta \in E_\alpha$ for each fixed $\delta$. $g_\delta$ converges to $h$ pointwise as $\delta \downarrow 0$. Moreover $|g_\delta - h|^2$ is dominated by $4\max\left(\sup_{[-1,1]} |h|, |\alpha|\right) = M \in \mathbb{L}^2\,[-1,1]$. By dominated convergence theorem $g_\delta \to h$ in $\mathbb{L}^2\,[-1,1]$. Thus $\exists \delta_0$ such that $|g_{\delta_0} - h|_{\mathbb{L}^2[-1,1]} < \frac{\epsilon}{2}$.

Thus $|g_{\delta_0} - f|_{\mathbb{L}^2[-1,1]} < \epsilon$. Hence $E_\alpha$ is dense in $\mathbb{L}^2\,[-1,1]$

b) Let $f_\alpha \in E_\alpha$ and $f_\beta \in E_\beta$. wlog let $\alpha < \beta$. By continuity of $f_\alpha$, $\exists \delta_\alpha$ such that $|x| < \delta_\alpha \implies f_\alpha < \alpha + \frac{(\beta-\alpha)}{4}$. By continuity of $f_\beta$, $\exists \delta_\beta$ such that $|x| < \delta_\beta \implies f_\beta > \beta - \frac{(\beta-\alpha)}{4}$. Thus on $|x| < \min(\delta_\alpha, \delta_\beta)$, $|f_\alpha - f_\beta| > \frac{|\beta-\alpha|}{4}$. Hence $|f_\alpha - f_\beta|_{\mathbb{L}^2[-1,1]} \geq \frac{|\beta-\alpha|}{4}\sqrt{\min(\delta_\alpha, \delta_\beta)} > 0$. Hence $f_\alpha \neq f_\beta$. Thus $E_\alpha \cap E_\beta = \{\phi\}$.

Let $l$ be a non zero linear functional. Then $\exists f \in \mathbb{L}^2\,[-1,1]$ such that $l\,(f) = c > 0$

We shall construct $g \in E_\alpha$ such that $l\,(g) > M$ for any $M$. Consider $(M+1)\,\frac{f}{c} \in \mathbb{L}^2\,[-1,1]$. Then by density of $E_\alpha$, $\exists h \in E_\alpha$ such that $\left|\frac{(M+1)f}{c} - h\right|_{\mathbb{L}^2} \leq \frac{1}{\|l\|}$. Then $l\,(h) = l\left(\frac{(M+1)f}{c}\right) + l\left(\frac{(M+1)f}{c} - h\right) = M+1+l\left(\frac{(M+1)f}{c} - h\right) \geq M+1-\|l\|\left|\frac{(M+1)f}{c} - h\right|_{\mathbb{L}^2[-1,1]} \geq M$.

$$\sup_{g\in E_\alpha} l\,(g) = \infty \tag{1}$$

$$\inf_{g\in E_\beta} l\,(g) \leq l\,(\beta) \leq |\beta|\,\|l\| \quad \text{(Since } \beta \in E_\beta\text{)} \tag{2}$$

From the above two relations, we see that **??** cannot hold for any non zero linear functional $l$.

Geometric Hahn Banach cannot be applied for two reasons, firstly both the sets are not closed and the intersection of their closures is everything since both sets are dense. Secondly, neither $E_\alpha$ or $E_\beta$ have an internal point. Let $h$ be the heavy side step function. Then for any $f \in E_\alpha$, $f + th \notin E_\alpha$ for any $t \neq 0$. Hence $E_\alpha$ does not have any interior point.

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7. Suppose that $\mathcal{P}$ is the space of all polynomials in one variable with real coefficients. Let the subset $A$ consist of polynomials with negative leading coefficients, and let the subset $B$ consist of polynomails with all non-negative coefficients. Show that $A$ and $B$ are disjoint convex subsets of $\mathcal{P}$. Further, show that there does not exist a nonzero linear functional $\ell$ on $\mathcal{P}$ such that

$$\ell(a) \leq \ell(b) \quad \forall a \in A\,, b \in B\,.$$

(Hint: assume that for some $C \in \mathbb{R}$, one has $\ell(a) \leq C \leq \ell(b)$, $a \in A$, $b \in B$;note that $0 \in B$ and that $C \leq 0$ and consider monomials to show that $C \geq 0$.
**Solution**: The fact that $A$ and $B$ are disjoint and convex is straight forward. Suppose that there exists some $C \in \mathbb{R}$ such that $\ell(a) \leq C \leq \ell(b)$ for all $a \in A$ and $b \in B$. Clearly $b = 0 \in B$, then $C \leq \ell(0) = 0$. Thus, $C \leq 0$. For each $n$, $-x^n \in A$, thus $\ell(-x^n) \leq C \leq 0$ which implies that $\ell(x^n) \geq 0$. Moreover, since $\ell \neq 0$, there exists some $n_0$ such that $\ell(x^{n_0}) > 0$. Now consider $p = -x^{n_0+1} + \alpha x^{n_0}$ where $\alpha > \ell(x^{n_0+1})/\ell(x^{n_0}$. Then $p \in A$ and hence $\ell(p) \leq C$. However,

$$C \geq \ell(p) = \ell(-x^{n_0+1} + \alpha x^{n_0}) = -\ell(x^{n_0+1}) + \alpha \ell(x^{n_0}) > 0\,.$$

Thus, $C > 0$ which is a contradiction.

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8. Construct two closed disjoint convex sets $K_1$ and $K_2$ in $\mathbb{R}^2$ that cannot be strictly separated, i..e there does not exist a bounded linear functional $\ell$ such that

$$\sup_{x \in K_1} \ell(x) < \inf_{y \in K_2} \ell(y)\,.$$

**Solution**: $K_1 = \{(x,y) : x \leq 0\}$ and $K_2 = \{(x,y) : xy \geq 1\}$.

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9. (a) Suppose that $T : X \to Y$ and $S : Y \to Z$ are bounded, where $X, Y, Z$ are Banach spaces. Show that $(ST)^* = T^* S^*$

   (b) Suppose that $S, T : X \to Y$ are bounded, where $X, Y$ are Banach spaces and suppose that $a, b \in \mathbb{R}$. Show that $(aS + bT)^* = aS^* + bT^*$.

   (c) Suppose that $T^{-1} : Y \to X$ exists and is bounded. Show that $(T^{-1})^* = (T^*)^{-1}$.

   **Solution**:

(a) Suppose that $\ell \in Z^*$ and $x \in X$, then

$$(ST^*\ell, x) := (\ell, STx) = (S^*\ell, Tx) = (T^*S^*\ell, x)$$

(b) Suppose $\ell \in Y^*$ and $x \in X$, then

$$((aS + bT)^*\ell, x) = (\ell, (aS + bT)x) = a\,(\ell, Sx) + b\,(\ell, Tx) = a\,(S^*\ell, x) + b\,(T^*\ell, x)\ .$$

(c) Suffices to show that $T^*\left(T^{-1}\right)^* = \left(T^{-1}\right)^* T^* = I$ which follows from the first part.

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