Problem set 3

Due date: Mar 5

February 26, 2018

1. Suppose that $\ell^1(\mathbb{N})$ is the space of sequences which are absolutely summable, i.e.

$$\{a_n\} \in \ell^1(\mathbb{N}) \quad \text{if} \quad \sum_{n=1}^{\infty} |a_n| < \infty \,.$$

   Suppose that $c_0(\mathbb{N})$ is the space of sequences that converge to 0, i.e.

$$\{a_n\} \in c_0(\mathbb{N}) \quad \text{if} \quad \lim_{n\to\infty} a_n = 0 \,.$$

   Show that $c_0(\mathbb{N})^* = \ell^1(\mathbb{N})$.

2. Suppose that $\ell^\infty(\mathbb{N})$ is the space of sequences which are bounded, i.e.

$$\{a_n\} \in \ell^\infty(\mathbb{N}) \quad \text{if} \quad \sup_n |a_n| < \infty \,.$$

   Show that $\ell^\infty(\mathbb{N})$ is not separable.

3. Suppose that $c \subset \ell^\infty(\mathbb{N})$ is the space of sequences that converge, i.e.

$$\{a_n\} \in c \quad \text{if} \quad \lim_{n\to\infty} a_n \quad \text{exists} \,.$$

   Show that c is a closed subspace.

4. (Optional) Recall that the dual of $C[0,1]$ is the space of Borel measures on the interval $[0,1]$. Construct a bounded linear functional in $(C[0,1])^*$ which does not attain its norm.

5. Let

$$\ell(f) = f(x_0) \,,$$

   denote a linear functional in $(C[0,1])^*$ where $0 < x_0 < 1$. Show that ℓ is a bounded and find the norm of ℓ.

6. Suppose that $X = L^2[-1,1]$. For each scalar α, let

$$E_\alpha := \{f \in C[-1,1], f(0) = \alpha\} \,.$$

   Show that

(a) Each E_α is convex and dense in X

(b) For $\alpha \neq \beta$, E_α, E_β are disjoint but there is no continuous functional on ℓ on X such that

$$\sup_{f \in E_\alpha} \ell(f) \leq \inf_{f \in E_\beta} \ell(f) \,.$$

Explain why geometric Hahn-Banach could not be employed

7. Suppose that $\mathcal{P}$ is the space of all polynomials in one variable with real coefficients. Let the subset A consist of polynomials with negative leading coefficients, and let the subset B consist of polynomails with all non-negative coefficients. Show that A and B are disjoint convex subsets of $\mathcal{P}$. Further, show that there does not exist a nonzero linear functional ℓ on $\mathcal{P}$ such that

$$\ell(a) \leq \ell(b) \quad \forall a \in A\,, b \in B\,.$$

(Hint: assume that for some $C \in \mathbb{R}$, one has $\ell(a) \leq C \leq \ell(b)$, $a \in A$, $b \in B$;note that $0 \in B$ and that $C \leq 0$ and consider monomials to show that $C \geq 0$.

8. Construct two closed disjoint convex sets K_1 and K_2 in $\mathbb{R}^2$ that cannot be strictly separated, i..e there does not exist a bounded linear functional ℓ such that

$$\sup_{x \in K_1} \ell(x) < \inf_{y \in K_2} \ell(y)\,.$$

9. (a) Suppose that $T : X \to Y$ and $S : Y \to Z$ are bounded, where X, Y, Z are Banach spaces. Show that $(ST)^* = T^* S^*$

   (b) Suppose that $S, T : X \to Y$ are bounded, where X, Y are Banach spaces and suppose that $a, b \in \mathbb{R}$. Show that $(aS + bT)^* = aS^* + bT^*$.

   (c) Suppose that $T^{-1} : Y \to X$ exists and is bounded. Show that $(T^{-1})^* = (T^*)^{-1}$.