Problem set 3
Due date: Mar 5
February 26, 2018

1. Suppose that \( \ell^1(\mathbb{N}) \) is the space of sequences which are absolutely summable, i.e.

\[
\{a_n\} \in \ell^1(\mathbb{N}) \quad \text{if} \quad \sum_{n=1}^{\infty} |a_n| < \infty.
\]

Suppose that \( c_0(\mathbb{N}) \) is the space of sequences that converge to 0, i.e.

\[
\{a_n\} \in c_0(\mathbb{N}) \quad \text{if} \quad \lim_{n \to \infty} a_n = 0.
\]

Show that \( c_0(\mathbb{N})^* = \ell^1(\mathbb{N}) \).

2. Suppose that \( \ell^\infty(\mathbb{N}) \) is the space of sequences which are bounded, i.e.

\[
\{a_n\} \in \ell^\infty(\mathbb{N}) \quad \text{if} \quad \sup_n |a_n| < \infty.
\]

Show that \( \ell^\infty(\mathbb{N}) \) is not separable.

3. Suppose that \( c \subset \ell^\infty(\mathbb{N}) \) is the space of sequences that converge, i.e.

\[
\{a_n\} \in c \quad \text{if} \quad \lim_{n \to \infty} a_n \text{ exists}.
\]

Show that \( c \) is a closed subspace.

4. (Optional) Recall that the dual of \( C[0, 1] \) is the space of Borel measures on the interval \([0, 1]\). Construct a bounded linear functional in \( (C[0, 1])^* \) which does not attain its norm.

5. Let

\[
\ell(f) = f(x_0),
\]

denote a linear functional in \( (C[0, 1])^* \) where \( 0 < x_0 < 1 \). Show that \( \ell \) is a bounded and find the norm of \( \ell \).

6. Suppose that \( X = L^2[-1, 1] \). For each scalar \( \alpha \), let

\[
E_\alpha := \{f \in C[-1, 1], f(0) = \alpha\}.
\]

Show that
(a) Each $E_\alpha$ is convex and dense in $X$

(b) For $\alpha \neq \beta$, $E_\alpha$, $E_\beta$ are disjoint but there is no continuous functional on $\ell$ on $X$ such that

$$\sup_{f \in E_\alpha} \ell(f) \leq \inf_{f \in E_\beta} \ell(f).$$

Explain why geometric Hahn-Banach could not be employed

7. Suppose that $\mathcal{P}$ is the space of all polynomials in one variable with real coefficients. Let the subset $A$ consist of polynomials with negative leading coefficients, and let the subset $B$ consist of polynomials with all non-negative coefficients. Show that $A$ and $B$ are disjoint convex subsets of $\mathcal{P}$. Further, show that there does not exist a nonzero linear functional $\ell$ on $\mathcal{P}$ such that

$$\ell(a) \leq \ell(b) \quad \forall a \in A, b \in B.$$ 

(Hint: assume that for some $C \in \mathbb{R}$, one has $\ell(a) \leq C \leq \ell(b)$, $a \in A$, $b \in B$; note that $0 \in B$ and that $C \leq 0$ and consider monomials to show that $C \geq 0$.

8. Construct two closed disjoint convex sets $K_1$ and $K_2$ in $\mathbb{R}^2$ that cannot be strictly separated, i.e. there does not exist a bounded linear functional $\ell$ such that

$$\sup_{x \in K_1} \ell(x) < \inf_{y \in K_2} \ell(y).$$

9. (a) Suppose that $T : X \rightarrow Y$ and $S : Y \rightarrow Z$ are bounded, where $X, Y, Z$ are Banach spaces. Show that $(ST)^* = T^*S^*$

(b) Suppose that $S, T : X \rightarrow Y$ are bounded, where $X, Y$ are Banach spaces and suppose that $a, b \in \mathbb{R}$. Show that $(aS + bT)^* = aS^* + bT^*$.

(c) Suppose that $T^{-1} : Y \rightarrow X$ exists and is bounded. Show that $(T^{-1})^* = (T^*)^{-1}$. 

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