Problem set 2

Due date: Feb 19

March 5, 2018

1. Suppose that T is a symmetric bounded operator. Then show that

$$\|T\| = \sup\{|(Tf, f)|, \quad \|f\| = 1\}.$$

Hint: You may assume the polarization identity

$$(Tf, g) = \frac{1}{4}[(T(f+g), f+g) - (T(f-g), f-g) + i(T(f+ig), f+ig) - i(T(f-ig), f-ig)]$$

Solution: Let $M. = \sup\{\{|(Tf, f)|, \quad \|f\| = 1\}$. Then clearly,

$$\begin{aligned} |(Tf, f)| &\leq \|Tf\| \cdot \|f\| \quad \text{(Cauchy Schwarz)} \\ &\leq \|T\| \cdot \|f\|^2 \quad \text{(Definition of operator norm)} \\ &\leq \|T\| \quad (\|f\| = 1). \end{aligned}$$

Thus $M \leq \|T\|$. To show the other direction, recall that

$$\|T\| = \sup\{|(Tf, g)|, \quad \|f\| = 1, \quad \|g\| = 1\}.$$

Note that, when T is symmetric (Th, h) is real for an $h \in \mathcal{H}$, since

$$(Th, h) = (h, T^*h) = (h, Th) = \overline{(Th, h)}.$$

Combining this with the polarization identity, we get

$$\begin{aligned} |\text{Re}(Tf, g)| &= \left| \frac{1}{4} \left[(T(f+g), f+g) - (T(f-g), f-g) \right] \right| \\ &\leq \frac{1}{4} \left[|(T(f+g), f+g) + (T(f-g), f-g)| \right] \quad \text{(Triangle inequality)} \\ &\leq \frac{1}{4} \left[M\|f+g\|^2 + M\|f-g\|^2 \right] \quad (|(Th, h)| \leq M\|h\|^2) \\ &\leq \frac{M}{4} \left[\|f\|^2 + \|g\|^2 + \|f\|^2 + \|g\|^2 \right] \quad \text{(Triangle inequality)} \\ &\leq M \end{aligned}$$

A simple rotation calculation shows that

$$\|T\| = \sup\{|(Tf, g)|, \|f\| = \|g\| = 1\} = \sup\{|\text{Re}(Tf, g)|, \|f\| = \|g\| = 1\}$$

Thus, we conclude that $\|T\| \leq M$, which completes the proof.

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2. Suppose that G is a compact set in $\mathbb{R}^n$. Suppose that

$$T[f](x) = \int_G K(x,y)f(y)dy\,,$$

where $K : G \times G \to \mathbb{R}$ is a continuous function for all $x, y \in G$ except for $x = y$. Furthermore, suppose that K satisfies

$$|K(x,y)| \le \frac{C}{|x-y|^\alpha}\,,$$

where $\alpha > 0$. Find the range of values of α for which the operator $T : \mathbb{L}^2(G) \to \mathbb{L}^2(G)$ is compact. Hint: Integral operators with continuous kernels are compact, and the norm limit of compact operators is compact.

Solution: Let

$$h(t) = \begin{cases} 1 & 1 \le t \\ 2t-1 & 1/2 \le t < 1 \\ 0 & 0 \le t < 1/2 \end{cases}.$$

Set

$$K_n(x,y) = K(x,y)h(m|x-y|)\,.$$

Let

$$T_m[f](x) = \int_G K_m(x,y)f(y)\,dy\,.$$

Then, T_m is compact since the kernel K_m is continuous and G is compact. We will now show that $T_m \to T$ in operator norm as long as $\alpha < n$.

$$\begin{aligned} |\left(T_m - T\right)[f](x)| &= |\int_G (K_m(x,y) - K(x,y))f(y)dy| \\ &\le \int_G |K_m(x,y) - K(x,y)f(y)dy \\ &\le \int_G \frac{C}{\|x-y\|^\alpha}\chi_{|x-y|\le\frac{1}{m}}|f(y)|\,dy\,. \end{aligned}$$

Here χ_A is the indicator function of the set A Then by using Young's inequality,

$$\|(T_m - T)[f]\| \le \left\|\frac{1}{|x|^\alpha}\chi_{|x|\le\frac{1}{m}}\right\|_{\mathbb{L}^1(G)} \|f\|_{\mathbb{L}^2(G)}\,,$$

which clearly converges to 0 as $m \to \infty$ if $\alpha < n$.

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3. Consider the operator $T : \mathbb{L}^2([0,1]) \to \mathbb{L}^2([0,1])$ defined by

$$T[f](t) = t \cdot f(t)$$

(a) Prove that T is a bounded linear operator with $T = T^*$, but that T is not compact

(b) However, show that T has no eigenvectors

The multiplication operator defined above is shown to have a critical role in the design of quadratures (see , for example).
Solution:
Boundedness of T

$$\|Tf\|_{\mathbb{L}^2}^2 = \int_0^1 |t|^2|f(t)|^2 dt \leq \int_0^1 |f(t)|^2 \quad (|t| < 1)$$
$$= \|f\|_{\mathbb{L}^2}^2 .$$

Thus, $\|T\| \leq 1$.
Adjointness of T

$$(Tf, g) = \int_0^1 tf(t) \cdot g(t)dt = \int_0^1 f(t) \cdot (tg(t)) = (f, T^*g) .$$

Thus, $T^*g = t \cdot g(t)$.
Non-compactness of T Consider the sequence $f_n(t) = \sin(2\pi nt)$. Then $\|f_n\|^2 = \frac{1}{2}$ and

$$\begin{aligned} \|Tf_n - Tf_m\|_{\mathbb{L}^2}^2 &= \int_0 (t\sin(2\pi nt) - t\sin(2\pi mt))^2 \, dt \\ &= \frac{1}{3} - \frac{1}{16\pi^2 n^2} - \frac{1}{16\pi^2 m^2} + \frac{1}{4\pi^2(n+m)^2} - \frac{1}{4\pi^2(n-m)^2} \not\to 0 \quad \text{as} \quad n, m \to \infty \end{aligned}$$

T has no eigenvectors. Let $\lambda \in \mathbb{C}$, then

$$Tf - \lambda f = 0 \implies (t - \lambda) \cdot f(t) = 0 .$$

Since $(t - \lambda) \neq 0$ almost everywhere, we conclude that f must be 0 almost everywhere and thus λ is not an eigenvalue.

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4. Let $\mathcal{H}$ be a Hilbert space with basis $\{e_k\}_{k=1}^{\infty}$. Verify that the operator T defined by

$$T(e_k) = \frac{e_{k+1}}{k} ,$$

is compact, but has no eigenvectors.
Solution: Compactness of T Let P_n be the projection operator onto the first n components and set $T_n = TP_n$. Clearly, T_n is a finite rank operator, since $\mathrm{Ran}(T_n) = \mathrm{span}\{e_1, e_2, \ldots e_{n+1}\}$. Then for all $\|f\| \leq 1$,

$$\|(T - TP_n)f\|^2 = \sum_{m=n+1}^{\infty} (f_{m+1}/m)^2 \leq \sum_{m=n+1}^{\infty} \frac{1}{m^2} .$$

Thus,

$$\|T - TP_n\| = \sup_{\|f\|=1} \|(T - TP_n)f\| \leq \sqrt{\sum_{m=n+1}^{\infty} \frac{1}{m^2}} \to 0$$

as $n \to \infty$. Thus, T is the norm limit of finite rank operators and hence is compact. T has no eigenvectors. Suppose $\lambda in \mathbb{C} \neq 0$, then consider

$$Tf - \lambda f = (-\lambda f_1, -\lambda f_2 + f_1, \ldots, -\lambda f_{n+1} + \frac{f_n}{n}, \ldots) .$$

If $Tf - \lambda f = 0$, then $f_{n+1} = f_n / n\lambda$ and $\lambda f_1 = 0$, from which we conclude that $f_n = 0$ for all n. If $\lambda = 0$, i.e. $Tf = (0, f_1, f_2/2, f_3/3, \ldots) = 0$, which implies again that $f = 0$. Thus T has no eigenvectors.

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5. Let $\mathcal{H}$ be a Hilbert space with basis $\{e_k\}_{k=1}^{\infty}$. Verify that the operator T defined by

$$T(e_k) = \lambda_k e_k ,$$

is compact if and only if $\lim_{k\to\infty} \lambda_k \to 0$.
Solution: Suppose T is compact, then λ_k are the eigenvalues of T and it follows from the spectral theorem that $\lambda_k \to 0$. Now suppose that $\lambda_k \to 0$. Then for any $\varepsilon > 0$, there exists N such that $|\lambda_n| \leq \varepsilon$ for all $n \geq N$. Let P_n denote the projection operator on to the basis $\{e_1, e_2 \ldots, e_n\}$. Then TP_n is finite rank for any n, and for any f and $n > N$

$$\|(T - TP_n)f\|^2 = \sum_{n=N+1} |\lambda_n|^2 |f_n|^2 \leq \varepsilon^2 \|f\|^2 .$$

Thus, for all $n > N$, we conclude that

$$\|T - TP_n\| \leq \varepsilon ,$$

from which we conclude that T is the norm limit of finite rank operators and hence T is compact.

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6. Let $\sigma(T)$ denote the spectrum of a compact operator $T : \mathcal{H} \to \mathcal{H}$. Show that $\lambda \in \sigma(T)$ if and only if $\overline{\lambda} \in \sigma(T^*)$.
Solution: Follows from

$$\dim(\mathcal{N}(\lambda I - T)) = \dim(\mathcal{N}(\overline{\lambda} I - T^*)) .$$

Thus, $\lambda \notin \sigma(T)$, if and only if $\mathcal{N}(\lambda I - T) = \{0\}$, if and only if, $\mathcal{N}(\overline{\lambda} I - T^*) = 0$, if and only if $\overline{\lambda} \notin \sigma(T^*)$.

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7. Let K be a Hilbert-Schmidt kernel which is real and symmetric, i.e. $K : [0,1] \times [0,1] \to \mathbb{R}$ satisfies $K(x,y) = K(y,x)$ and $K \in \mathbb{L}^2([0,1] \times [0,1])$. Let $T : \mathbb{L}^2([0,1]) \to \mathbb{L}^2([0,1])$ be defined by

$$T[f](x) = \int_0^1 K(x,y) f(y) dy .$$

Let $\phi_k(x)$ be the eigenvectors (with eigenvalues λ_k) that diagonalize T. Then:

(a) $\sum_k |\lambda_k|^2 < \infty$

(b) $K(x,y) = \sum_{k=1}^{\infty} \lambda_k \phi_k(x)\phi_k(y)$

(c) Suppose $\tilde{T}$ is an operator which is compact and symmetric. Then $\tilde{T}$ is of Hilbert-Schmidt type if and only if $\sum_n |\lambda_n|^2 < \infty$, where $\{\lambda_n\}$ are the eigenvalues of $\tilde{T}$ counted according to their multiplicities

Solution:

(a) Follows from part b and the fact that K is of Hilbert-Schmidt type

(b) Let ϕ_j, $j = 1, 2, \ldots$ be an orthogonal basis for $\mathbb{L}^2([0,1])$, then we know that $\phi_j(x) \cdot \phi_\ell(y)$, $j, \ell = 1, 2, \ldots$ forms an orthogonal basis for $\mathbb{L}^2[0,1]$ and that

$$K(x,y) = \sum_{j,\ell=1}^{\infty} a_{j,\ell}\phi_j(x)\phi_\ell(y)\,,$$

with

$$\sum_{j,\ell} |a_{j,\ell}|^2 < \infty\,.$$

Since ϕ_k is an eigenvalue of the operator T with eigenvalue λ_k, we have

$$\begin{aligned} \lambda_k\phi_k(x) &= \int_0^1 K(x,y)\phi_k(y)\,dy \\ &= \int_0^1 \sum_{j,\ell} a_{j,\ell}\phi_j(x)\phi_\ell(y)\cdot\phi_k(y)dy \\ &= \sum_{j=1}^{\infty} a_{j,k}\phi_j(x) \quad (\text{Since } \phi_\ell(y) \perp \phi_j(y)) \end{aligned}$$

Taking inner products with $\phi_\ell(x)$ and using the orthogonality of ϕ_j's, we conclude that $a_{j,k} = 0$ is $j \neq k$ and $a_{j,k} = \lambda_k$ if $j = k$.

(c) For the third part define

$$K_n(x,y) = \sum_{\ell=1}^{n} \lambda_k\phi_k(x)\phi_k(y)\,.$$

Here $\phi_k(x)$ are the eigenvectors associated with eigenvalue λ_k. Since, the $\lambda_k's$ are square summable, K_n is a Cauchy sequence in $\mathbb{L}^2[0,1]\times[0,1]$. Thus, $K_n \to K(x,y)$ in $\mathbb{L}^2[0,1]\times[0,1]$. Define $T_n = P_n\tilde{T}$, where P_n is the projection onto the first n eigenvectors. Then $T_n f = \int_0^1 K_n(x,y)f(y)\,dy$. Moreover, since $\lambda_k \to 0$, $T_n \to \tilde{T}$ in norm. Moreover, a simple application of Holder shows that

$$\|T_n - \tilde{T}\| \leq \|K_n - K\|_{\mathbb{L}^2[0,1]\times[0,1]}\,.$$

Thus, $\tilde{T}$ is the integral operator with kernel K.

8. Let $\mathcal{H}$ be a Hilbert space.

   (a) If $T_1, T_2 : \mathcal{H} \to \mathcal{H}$ are compact symmetric operators which commute, i.e. ($T_1T_2 = T_2T_1$), show that they can be diagonalized simultaneously. In other words, there exists an orthonormal basis for $\mathcal{H}$ which consists of eigenvectors for both T_1 and T_2.

   (b) A linear operator on $\mathcal{H}$ is normal if $TT^* = T^*T$. Prove that if T is normal and compact, then T can be diagonalized.

   (c) If U is unitary, and $U = \lambda I - T$, where T is compact, then U can be diagonalized.

   Solution:

   (a) Suppose λ_i are the collection of eigenvalues of T_1 and $E_{\lambda_i}(T_1)$ are the corresponding eigenspaces. Then we will show that an orthogonal collection in E_{λ_i} also are eigenvectors of T_2. Suppose that $f_1, f_2, \dots f_n$ forms a basis for E_λ, then

   $$T_1 f_j = \lambda_1 f_j \,.$$

   Thus,

   $$T_1 T_2 f_j = T_2 T_1 f_j = \lambda T_2 f_j \,,$$

   i.e., $T_2 f_j \in E_\lambda(T_1)$, i.e.

   $$T_2 f_j = \sum_{i=1}^{n} \alpha_{i,j} f_i \,.$$

   Thus, $T_2 : E_\lambda \to E_\lambda$ can be represented as an $n \times n$ matrix with entries $\alpha_{i,j}$. From the symmetry of T_2, it follows that $\alpha_{i,j} = \alpha_{j,i}$ and thus, the orthogonal matrix has a collection of orthogonal eigenvectors of the mapping T_2. This, shows that every eigenvector of T_1 with eigenvalue not equal to 0 is also an eigenvector of T_2. For $\lambda = 0$, a similar proof shows that $T_2 : \mathcal{N}(T_1) \to \mathcal{N}(T_1)$ and it follows from the spectral theorem, that there exists an orthogonal basis of $\mathcal{N}(T_1)$ which are the eigenvectors of T_2 too.

   (b) For normal matrices as well, it follows from the polarization identity that

   $$\|T\| = \sup\{\ |(f, Tf)|, \quad \|f\| = 1\} \,.$$

   (c) Since U is unitary $UU^* = U^*U = I$, from which it follows that $TT^* = T^*T$. From the previous part, T is diagonalizable, and a simple calculation shows that eigenvectors v_i of T associated with eigenvalue λ_i are also eigenvectors of U with eigenvalue $\lambda - \lambda_i$.

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9. Fredholm theory for non-zero index operators. An operator R is called a regularizer of an operator K if R is bounded and $RK = I - A_\ell$ and $KR = I - A_r$, where A_ℓ, A_r are compact.

(a) Suppose that $K : \mathcal{H} \to \mathcal{H}$, and R is a regularizer of K, then $\dim\{\mathcal{N}(K)\} < \infty$ and $\dim\{\mathcal{N}(R)\} < \infty$

(b) If $RK = I - A$, where A is compact, show that $\phi - A\phi = Rf$ has a solution for every $f \in \mathcal{N}(K^*)^\perp$

(c) Now further assume that $N(I - A) = \{0\}$. Suppose that $S = (I - A)^{-1}$. Show that $\operatorname{Ran}((I - KSR)) \subset \mathcal{N}(R)$ and that $\operatorname{Ran}((I - KSR)^*) \subset \mathcal{N}(K^*)$. Combine the previous result and these results to show that $\phi = SRf$ also satisfies $K\phi = f$ as long as $f \in N(\mathcal{K}^*)^\perp$.

(d) (optional, no extra credit) Show that Ran(K) $= \mathcal{N}(K^*)^\perp$ for any operator K which has a regularizer

Solution:

(a) $\mathcal{N}(K) \subset \mathcal{N}(RK) = \mathcal{N}(I - A_\ell)$. $\dim \mathcal{N}(I - A_\ell) < \infty$ implies that $\dim \mathcal{N}(K) < \infty$. We can think of K as a regularizer of R as well, and hence $\dim \mathcal{N}(R)$ is also finite.

(b) Suppose $g \in \mathcal{N}(K^* R^*)$, then

$$K^* R^* g = 0 \implies R^* g \in \mathcal{N}(K^*)\,.$$

Thus, for every $f \in \mathcal{N}(K^*)^\perp$,

$$(Rf, g) = (f, R^* g) = 0\,.$$

Thus, $Rf \in \mathcal{N}(K^* R^*)^\perp$ and hence $Rf \in \operatorname{Ran}(RK)$.

(c) Suppose $\phi \in \operatorname{Ran}(I - KSR)$, then $\exists g \in \mathcal{H}$ such that

$$\phi = (I - KSR)g\,.$$

Then,

$$R\phi = Rg - RKSRg = Rg - (I-A)\cdot(I-A)^{-1}Rg = 0 \quad (RK = I - A \quad \text{and} S = (I-A)^{-1})\,.$$

Thus, $\phi \in \mathcal{N}(R)$ and that $\operatorname{Ran}(I - KSR) \subset \mathcal{N}(R)$. The other result follows in a similar manner. Since $I - A$ is injective, $I - A$ has a bounded inverse. Set $\phi = SRf$. Then,

$$f - K\phi = f - KSRf \in \operatorname{Ran}(I - KSR) \implies f - KSRf \in \mathcal{N}(R)$$

. A similar argument shows that $\operatorname{Ran}(I - R^* S^* K^*) \subset \mathcal{N}(K^*)$. From the first part, $\dim(\mathcal{N}(R)) < \infty$, and let ϕ_i, $i = 1, 2, \ldots N$, be an orthogonal basis for $\mathcal{N}(R)$. Then

$$f - KSRf = \sum_{i=1}^{N} c_i \phi_i\,,$$

where

$$c_i = (f - KSRf, \phi_i) = (f, \phi_i - R^* S^* K^* \phi_i) = 0\,,$$

where the last equality follows from the fact that $\phi_i - R^* S^* K^* \in \operatorname{Ran}(I - R^* S^* K^*) \subset \mathcal{N}(K^*)$ and $f \in \mathcal{N}(K^*)^\perp$.
