Problem set 1
Due date: Feb 5
February 14, 2018

1. If \( \{h_n\} \) is a sequence in a Hilbert space \( \mathcal{H} \) such that \( \sum_n \|h_n\| < \infty \), then show that \( h_n \) converges.
   
   **Solution:** Since \( \sum_n \|h_n\| < \infty \), we conclude that \( \|h_n\| \to 0 \), and thus \( h_n \to 0 \).

2. Suppose that \( E \) is a linear subspace of a Hilbert space \( \mathcal{H} \), then show that the closure of \( E \) is also a linear subspace.
   
   **Solution:** Suppose that \( x, y \in \overline{E} \), then there exist sequences \( \{x_n\}_{n=1}^\infty, \{y_n\}_{n=1}^\infty \in E \) such that \( x_n \to x \) and \( y_n \to y \). For any \( c_1, c_2 \in \mathbb{F} \), then \( \{c_1x_n + c_2y_n\}_{n=1}^\infty \in E \), since \( E \) is a linear subspace. Moreover, \( c_1x_n + c_2y_n \to c_1x + c_2y \). Thus, \( c_1x + c_2y \in \overline{E} \).

3. Suppose that \( E \) is a subspace of a Hilbert space \( \mathcal{H} \), then show that \( (E^\perp)^\perp \) is the closure of the span of elements in \( E \), i.e.

   \[
   (E^\perp)^\perp = \left\{ \sum_{j=1}^N c_j f_j , \ f_j \in E \right\}
   \]

   **Solution:** Since \( E^\perp \) is closed for any subspace \( E \), it suffices to show that finite linear combinations of elements in \( E \) are in \( (E^\perp)^\perp \). Suppose that \( f_j \in E, j = 1, 2, \ldots N \), and suppose \( c_j \in \mathbb{F}, j = 1, 2, \ldots N \). Then \( f = \sum_{j=1}^N c_j f_j \in (E^\perp)^\perp \), since for any \( g \in E^\perp \),

   \[
   (f, g) = \left( \sum_{j=1}^N c_j f_j , g \right) = \sum_{j=1}^N c_j (f_j, g) \ (\text{Linearity of inner product}) = 0 \ (\text{Since } g \in E^\perp, f_j \in E \implies (f_j, g) = 0)
   \]
4. Suppose that $\mathcal{H} = \ell^2(\mathbb{N})$.

(a) Show that if $\{a_n\} \in \mathcal{H}$, then the power series $\sum_{n=1}^{\infty} a_n z^n$ has radius of convergence at least 1.

(b) For $\lambda < 1$, show that $L(\{a_n\}) := \sum_{n=1}^{\infty} a_n \lambda^n$ is a bounded linear functional.

(c) Find the element $h_0 \in \mathcal{H}$ such that $L(h) = (h, h_0)$ and find $\|L\|$. 

**Solution:**

a) If $\{a_n\} \in \mathcal{H}$, then $a_n$ is a bounded sequence, i.e. $|a_n| \leq M$. Thus, it follows from the Weierstrass-M test that, for all $|z| = \rho < 1$, $\sum_{n=1}^{\infty} a_n z^n$ converges since $|a_n z^n| \leq M \rho^n$. Thus, radius of convergence is at least 1.

b) This follows from problem 7, with $\alpha_n = \lambda^n$.

c) $h_0 = \{\lambda^n\}$.

5. Let $\mathcal{H}_1 = L^2([-\pi, \pi])$ be the Hilbert space of functions $F(e^{i\theta})$ on the unit circle with the inner product

$$\langle F, G \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta}) G(e^{i\theta}) d\theta.$$

Let $\mathcal{H}_2$ be the space $L^2(\mathbb{R})$. Using the mapping

$$x \rightarrow \frac{i - x}{i + x}$$

of $\mathbb{R}$ to the unit circle, show that:

a) The correspondence $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ given by

$$U[F] = \frac{1}{\pi^{1/2}(i + x)} F\left(\frac{i - x}{i + x}\right)$$

is a unitary mapping.

b) As a result show that

$$\left\{ \frac{1}{\pi^{1/2}(i + x)} \left(\frac{i - x}{i + x}\right)^n \right\}_{n=-\infty}^{\infty}$$

is an orthonormal basis of $L^2(\mathbb{R})$.

**Solution:** $U(F) = \frac{1}{\sqrt{\pi}} \frac{1}{i + x} F\left(\frac{i - x}{i + x}\right)$ is a linear function in $F$. We will show that $U(F)$ is norm preserving.
\[|U(F)|^2_{L_2(\mathbb{R})} = \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{1}{|i+x|^2} \left| F \left( \frac{i-x}{i+x} \right) \right|^2 \, dx \]
\[= \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{1+x^2} \left| F \left( \frac{i-x}{i+x} \right) \right|^2 \, dx \]
\[= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| F \left( \frac{i - \tan \left( \frac{\theta}{2} \right)}{i + \tan \left( \frac{\theta}{2} \right)} \right) \right|^2 \, d\theta \quad \text{(Making the change of variable } x = \tan \left( \frac{\theta}{2} \right)) \]
\[= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| F \left( \frac{\cos \left( \frac{\theta}{2} \right) i - \sin \left( \frac{\theta}{2} \right)}{i \cos \left( \frac{\theta}{2} \right) + \sin \left( \frac{\theta}{2} \right)} \right) \right|^2 \, d\theta \]
\[= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| F \left( \frac{i e^{i\frac{\theta}{2}}}{i e^{-i\frac{\theta}{2}}} \right) \right|^2 \, d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| F \left( e^{i\theta} \right) \right|^2 \, d\theta \]
\[= |F|^2_{H_1} \]

Therefore \(U\) is norm preserving and hence \(1 - 1\).

Consider the mapping \(\overline{U} : H_2 \to H_1\) defined by \(\overline{U} f = f \left( \tan \left( \frac{\theta}{2} \right) \right) \left( i + \tan \left( \frac{\theta}{2} \right) \right) \sqrt{\pi}\).

By a similar calculation as above, we can show that \(|\overline{U} f|_{H_1} = |f|_{H_2}|\).

Let \(f \in H_2\) and \(F \in H_1\).

Claim: \(\overline{U} \circ U \left( F \right) = F\)

Proof:

\[\overline{U} \circ U \left( F \right) = \frac{1}{\sqrt{\pi}} \frac{1}{i + \tan \left( \frac{\theta}{2} \right)} F \left( \frac{i - \tan \left( \frac{\theta}{2} \right)}{i + \tan \left( \frac{\theta}{2} \right)} \right) \left( i + \tan \left( \frac{\theta}{2} \right) \right) \sqrt{\pi} \]
\[= F \left( e^{i\theta} \right) \]

Similarly, we can show that \(U \circ \overline{U} \left( f \right) = f\) and hence \(\overline{U} = U^{-1}\) and thus \(U\) must be onto.

Combining all of these, we see that \(U\) is a unitary correspondence between \(H_1\) and \(H_2\).

We know that \(\{\phi_n\}_{n=-\infty}^{\infty}\) where \(\phi_n \left( x \right) = \frac{1}{\sqrt{2\pi}} e^{inx}\) is an orthogonal basis of \(L^2 \left( [-\pi, \pi] \right)\). This means that \(F_n \left( e^{i\theta} \right) = e^{i\theta n}\) is an orthonormal basis for \(H_1\). Then the claim is that \(f_n \left( x \right) = UF_n \left( \frac{i+ix}{i-x} \right)^n \frac{1}{i+x} \) forms an orthonormal basis for \(L^2 \left( \mathbb{R} \right)\)

\[|F_n|_{H_1} = |UF_n|_{H_2} \quad (F_n, F_m)_{H_1} = (UF_n, UF_m)_{H_2} \]

Hence \(\{f_n\}\) is orthonormal in \(H_2\). Let \(f \in H_2\). Then \(\overline{U} f \in H_1\). Suppose \(f\) is orthogonal to all the basis vectors \(f_n\) then \((f, f_n)_{H_2} = 0\) for all \(n\).
(f, f_n)_{L^2} = (\mathcal{U} f, F_n)_{L^1} = 0

Hence \( \mathcal{U} f \equiv 0 \) since it is orthogonal to all basis elements \( F_n \). \( \therefore \) \( U \circ \mathcal{U} f \equiv 0 \) and hence \( f \equiv 0 \)

6. Prove that the operator \( T : L^2[0, \infty] \rightarrow L^2[0, \infty] \)

\[
T[f](x) = \frac{1}{\pi} \int_0^\infty \frac{f(y)}{x+y} \, dy
\]

is bounded operator with norm \( \|T\| \leq 1 \).

**Solution:** Use problem 4, practice problem set 1, with \( w(x) = \frac{1}{\sqrt{x}} \).

7. Suppose that the multiplication operator \( A : \ell^2(\mathbb{N}) \rightarrow \ell^2(\mathbb{N}) \) is defined via \( Ae_n = \alpha_n e_n \) where \( \{e_i\}_{i=1}^\infty \) are the standard coordinate vectors and \( \alpha_n \in \mathbb{R} \). Then show that \( A \) is bounded if and only if \( \sup_n |\alpha_n| \leq M \).

**Solution:** \( \sup_n |\alpha_n| \leq M \implies A \) is bounded.

If \( f \in \mathcal{H} \), then \( f = \sum_n (f, e_n) e_n \) and \( Af = \sum_n \alpha_n (f, e_n) e_n \). By Parseval,

\[
\|f\|^2 = \sum_{n=1}^\infty |(f, e_n)|^2,
\]

and

\[
\|Af\|^2 = \sum_{n=1}^\infty |\alpha_n (f, e_n)|^2 \leq M^2 \sum_n \|(f, e_n)\|^2 = M^2 \|f\|^2.
\]

Thus, \( A \) is bounded.

\( A \) is bounded \( \implies \sup_n \|\alpha_n\| < \infty \).

Suppose not. Then there exists a subsequence \( n_k, k = 1, 2, \ldots \), such that \( |\alpha_{n_k}| \geq k \). \( A \) being bounded implies that there exists an \( M < \infty \) such that \( \|Af\| \leq M\|f\| \) for any \( f \in \mathcal{H} \). However \( \|Ae_{n_k}\| \geq k \|e_{n_k}\| \) holds for any \( k \) which is a contradiction.

8. Suppose that \( \mathcal{K} : L^2([0, 1]) \rightarrow L^2([0, 1]) \) is defined by

\[
\mathcal{K}[f] = \int_0^1 k(x, y) f(y) \, dy,
\]
where \( k(x, y) \in L^2([0, 1] \times [0, 1]) \). Show that \( \mathcal{K} \) is a bounded linear operator.

**Solution:**

\[
|\mathcal{K}[f](x)| = \left| \int_0^1 k(x, y) f(y) dy \right|
\leq \int_0^1 |k(x, y) f(y)| dy
\leq \sqrt{\left( \int_0^1 |k(x, y)|^2 dy \right) \cdot \|f\|_{L^2[0,1]}^2} \quad \text{(Hölder inequality)}
\]

\[
\therefore \int_0^1 \|\mathcal{K}[f](x)\|^2 dx \leq \int_0^1 \int_0^1 |k(x, y)|^2 dy dx \cdot \|f\|_{L^2[0,1]}^2
\]

\[
\therefore \|\mathcal{K}[f]\|_{L^2[0,1]} \leq \sqrt{\left( \int_0^1 \int_0^1 |k(x, y)|^2 dy dx \right) \cdot \|f\|_{L^2[0,1]}^2}
\]

9. Give two examples of linear subspaces of \( L^2(\mathbb{R}) \) which are not closed and find their closure.

**Solution:** There are many options here, for example, \( C^k \) functions, i.e. functions which have \( k \) continuous derivatives, which are compactly supported are both linear subspaces and dense in \( L^2(\mathbb{R}) \) for any \( k \).

10. Suppose that \( P_1 \) and \( P_2 \) are orthogonal projections onto subspaces \( S_1 \) and \( S_2 \). Show that \( P_2 P_1 \) is an orthogonal projection if and only if \( P_1 \) and \( P_2 \) commute, i.e. \( P_1 P_2 = P_2 P_1 \) and in this case \( P_2 P_1 \) projects onto \( S_2 \cap S_1 \). Give an example of two projection operators which do not commute.

**Solution:** From exercise 1 in the practice problem set, it is clear that \( P_j = P_j^* \) and \( P_j^2 = P_j \) for \( j = 1, 2 \). Suppose that \( P_1 P_2 \) is an orthogonal projection. Then \( P_1 P_2 = (P_1 P_2)^* = P_2^* P_1^* = P_2 P_1 \).

Now suppose that \( P_1 P_2 = P_2 P_1 \), then for all \( f \in \mathcal{H} \)

\[
(P_1 P_2 f, f - P_1 P_2 f) = (P_2 f, P_1^* f - P_1^* P_1 P_2 f)
= (P_2 f, P_1 f - P_2^2 P_2 f)
= (P_2 f, P_1 f - P_1 P_2 f)
= (P_2 f, P_1 f - P_2 P_1 f)
= (P_2 f, (I - P_2) P_1 f)
= 0 \quad \text{(since } P_2 f \in S_2 \text{ and } (I - P_2) P_1 f \in S_2^\perp \text{)}
\]
11. Let $\mathcal{H} = \mathbb{L}^2(\mathbb{R})$. Let $\mathcal{F}: \mathcal{H} \to \mathcal{H}$ be the Fourier transform

$$\mathcal{F}[f](x) = \int_{-\infty}^{\infty} e^{i2\pi xy} f(y) \, dy.$$ 

Then it is well known that $\mathcal{F}$ is a unitary map with the inverse

$$\mathcal{F}^{-1}[f](x) = \int_{-\infty}^{\infty} e^{-i2\pi xy} f(y) \, dy.$$ 

Let $f * g$ denote the convolution operator

$$f * g(x) = \int_{-\infty}^{\infty} f(x - y)g(y) \, dy.$$ 

Further, it is also known that

$$\mathcal{F}[fg](x) = \mathcal{F}[f] * \mathcal{F}[g],$$

and

$$\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g].$$

(a) Let $\chi_A(x)$ denote the indicator function of the set $A$, i.e. $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise. Suppose $k_0 > 0$. Show that

$$\mathcal{F}[\chi_{[-k_0,k_0]}] = \frac{\sin(2\pi k_0 x)}{\pi x}.$$ 

(b) Let $K(x) = \mathcal{F}[\chi_{[-k_0,k_0]}](x)$. Show that

$$\int_{-\infty}^{\infty} K(x - z)K(z - y) \, dz = K(x - y).$$

(c) Let $\mathcal{K}: \mathbb{L}^2(\mathbb{R}) \to \mathbb{L}^2(\mathbb{R})$ denote the operator defined by

$$\mathcal{K}[f](x) = \int_{-\infty}^{\infty} K(x - y)f(y) \, dy.$$ 

Show that $\mathcal{K}$ is a bounded operator.

(d) Use part (b) to show that $\mathcal{K}$ is a projection operator in the following sense, $\mathcal{K}[\mathcal{K}[f]] = \mathcal{K}[f]$

(e) Let $\mathcal{H}_0 \subset \mathcal{H}$ denote the subspace defined by:

$$f \in \mathcal{H}_0 \text{ if } \mathcal{F}[f](x) = 0 \quad \forall |x| > k_0.$$ 

Show that $\mathcal{H}_0$ is a closed linear subspace. $\mathcal{H}_0$ is the subspace of band-limited functions with band-limit $k_0$.

(f) Show that $\mathcal{K}$ is the projection operator onto $\mathcal{H}_0$. 
Solution: a)

\[ \mathcal{F}[\chi_{[-k_0, k_0]}] = \int_{-\infty}^{\infty} e^{i2\pi xy} \chi_{[-k_0, k_0]}(y) dy = \int_{-k_0}^{k_0} e^{i2\pi xy} dy = \frac{\sin(2\pi k_0 x)}{\pi x} \]

b) A simple calculation shows that

\[ (f, g) = (\mathcal{F}^{-1}[f], \mathcal{F}^{-1}g), \]

\[ \mathcal{F}^{-1}[K(x - z)](\xi) = e^{-2\pi i \xi x} \chi_{[-k_0, k_0]}(\xi), \]

and

\[ \mathcal{F}^{-1}[K(z - y)](\xi) = e^{-2\pi i \xi y} \chi_{[-k_0, k_0]}(\xi), \]

Combining these three results, we get

\[ \int_{-\infty}^{\infty} K(x - z) \cdot K(z - y) = \int_{-\infty}^{\infty} K(x - z) \cdot \overline{K(z - y)} dz \quad \text{K is real} \]

\[ = \int_{-\infty}^{\infty} \mathcal{F}^{-1}[K(x - z)](\xi) \cdot \overline{\mathcal{F}^{-1}[K(z - y)](\xi)} d\xi \]

\[ = \int_{-\infty}^{\infty} e^{-2\pi i \xi (x - y)} \chi_{[-k_0, k_0]}(\xi) d\xi \]

\[ = K(x - y) \]

c) We first note that

\[ \mathcal{F}[\mathcal{K}[f]] = \mathcal{F}[K * f] = \mathcal{F}[K] \cdot \mathcal{F}[f] = \chi_{[-k_0, k_0]}(x) \cdot \mathcal{F}[f](x). \quad (1) \]

Since, the Fourier transform is an isometry, we have

\[ \|\mathcal{K}[f]\| = \|\mathcal{F}[\mathcal{K}[f]]\| = \|\chi_{[-k_0, k_0]}(x) \cdot \mathcal{F}[f](x)\| \leq \|\mathcal{F}[f]\| = \|f\|. \]

Thus \( \mathcal{K} \) is a bounded operator with \( \|\mathcal{K}\| \leq 1 \).

d) Follows from part b.

e) Suppose \( f_n \to f \) in \( L^2(\mathbb{R}) \), then \( \mathcal{F}[f_n] \to \mathcal{F}[f] \) in \( L^2(\mathbb{R}) \). Since \( \mathcal{F}[f_n](x) = 0 \) for all \( n \) and all \( |x| > k_0 \), we conclude that \( \mathcal{F}[f](x) = 0 \) for almost every \( x \) such that \( |x| > k_0 \).
f) From equation 1, it follows that \( \mathcal{K}[f] \in \mathcal{H}_0 \) for all \( f \in \mathcal{H} \). Moreover

\[ (\mathcal{K}[f], f - \mathcal{K}[f]) = (\mathcal{F}[\mathcal{K}[f]], \mathcal{F}[f - \mathcal{K}[f]]) = \int_{-\infty}^{\infty} \chi_{[-k_0, k_0]}(x) \cdot \mathcal{F}[f](x) \cdot (1 - \chi_{[-k_0, k_0]}) \mathcal{F}[f](x) = 0 \]