Problem set 1

Due date: Feb 5

February 14, 2018

1. If $\{h_n\}$ is a sequence in a Hilbert space $\mathcal{H}$ such that $\sum_n \|h_n\| < \infty$, then show that h_n converges.
Solution: Since $\sum_n \|h_n\| < \infty$, we conclude that $\|h_n\| \to 0$, and thus $h_n \to 0$.

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2. Suppose that E is a linear subspace of a Hilbert space $\mathcal{H}$, then show that the closure of E is also a linear subspace

Solution: Suppose that $x, y \in \overline{E}$, then there exist sequences $\{x_n\}_{n=1}^{\infty}, \{y_n\}_{n=1}^{\infty} \in E$ such that $x_n \to x$ and $y_n \to y$. For any $c_1, c_2 \in \mathbb{F}$, then $\{c_1 x_n + c_2 y_n\}_{n=1}^{\infty} \in E$, since E is a linear subspace. Moreover, $c_1 x_n + c_2 y_n \to c_1 x + c_2 y$. Thus, $c_1 x + c_2 y \in \overline{E}$.

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3. Suppose that E is a subspace of a Hilbert space $\mathcal{H}$, then show that $(E^{\perp})^{\perp}$ is the closure of the span of elements in E, i.e.

$$(E^{\perp})^{\perp} = \overline{\left\{ \sum_{j=1}^{N} c_j f_j \,, \quad f_j \in E \right\}}$$

Solution: Since $E^{\perp}$ is closed for any subspace E, it suffices to show that finite linear combinations of elements in E are in $(E^{\perp})^{\perp}$. Suppose that $f_j \in E$, $j = 1, 2, \ldots N$, and suppose $c_j \in \mathbb{F}$, $j = 1, 2, \ldots N$. Then $f = \sum_{j=1}^{N} c_j f_j \in (E^{\perp})^{\perp}$, since for any $g \in E^{\perp}$,

$$\begin{aligned}
(f, g) &= \left(\sum_{j=1}^{N} c_j f_j, g \right) \\
&= \sum_{j=1}^{N} c_j (f_j, g) \quad \text{(Linearity of inner product)} \\
&= 0 \quad (\text{Since } g \in E^{\perp},\ f_j \in E \implies (f_j, g) = 0)
\end{aligned}$$

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4. Suppose that $\mathcal{H} = \ell^2(\mathbb{N})$.

   (a) Show that if $\{a_n\} \in \mathcal{H}$, then the power series $\sum_{n=1}^{\infty} a_n z^n$ has radius of convergence at least 1

   (b) For $\lambda < 1$, show that $L(\{a_n\}) := \sum_{n=1}^{\infty} a_n \lambda^n$ is a bounded linear functional

   (c) Find the element $h_0 \in \mathcal{H}$ such that $L(h) = (h, h_0)$ and find $\|L\|$

Solution: a) If $\{a_n\} \in \mathcal{H}$, then a_n is a bounded sequence, i.e. $|a_n| \leq M$. Thus, it follows from the Weierstrass-M test that, for all $|z| = \rho < 1$, $\sum_{n=1}^{\infty} a_n z^n$ converges since $|a_n z^n| \leq M\rho^n$. Thus, radius of convergence is at least 1.
b) This follows from problem 7, with $\alpha_n = \lambda^n$
c) $h_0 = \{\lambda^n\}$.

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5. Let $\mathcal{H}_1 = \mathbb{L}^2([-\pi, \pi])$ be the Hilbert space of functions $F(e^{i\theta})$ on the unit circle with the inner product

$$(F, G) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta}) \overline{G(e^{i\theta})} d\theta \,.$$

Let $\mathcal{H}_2$ be the space $\mathbb{L}^2(\mathbb{R})$. Using the mapping

$$x \to \frac{i - x}{i + x}$$

of $\mathbb{R}$ to the unit circle, show that:
a) The correspondence $U : \mathcal{H}_1 \to \mathcal{H}_2$ given by

$$U[F] = \frac{1}{\pi^{1/2}(i + x)} F(\frac{i - x}{i + x})$$

is a unitary mapping.
b) As a result show that

$$\left\{ \frac{1}{\pi^{1/2}(i + x)} \left(\frac{i - x}{i + x} \right)^n \right\}_{n=-\infty}^{\infty}$$

is an orthonormal basis of $\mathbb{L}^2(\mathbb{R})$.

Solution: $U\left(F\right) = \frac{1}{\sqrt{\pi}} \frac{1}{(i+x)} F\left(\frac{i-x}{i+x}\right)$ is a linear function in F. We will show that $U\left(F\right)$ is norm preserving

$$
\begin{aligned}
|U(F)|^2_{\mathbb{L}^2(\mathbf{R})} &= \int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{|i+x|^2}\left|F\left(\frac{i-x}{i+x}\right)\right|^2 dx \\
&= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{1+x^2}\left|F\left(\frac{i-x}{i+x}\right)\right|^2 dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F\left(\frac{i-\tan\left(\frac{s}{2}\right)}{i+\tan\left(\frac{s}{2}\right)}\right)\right|^2 ds \quad \left(\text{Making the change of variable } x=\tan\left(\tfrac{s}{2}\right)\right) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F\left(\frac{\cos\left(\frac{s}{2}\right)i-\sin\left(\frac{s}{2}\right)}{i\cos\left(\frac{s}{2}\right)+\sin\left(\frac{s}{2}\right)}\right)\right|^2 ds \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F\left(\frac{ie^{i\frac{s}{2}}}{ie^{-i\frac{s}{2}}}\right)\right|^2 ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|F\left(e^{is}\right)\right|^2 ds \\
&= |F|^2_{\mathcal{H}_1}
\end{aligned}
$$

Therefore U is norm preserving and hence $1-1$.

Consider the mapping $\overline{U} : \mathcal{H}_2 \to \mathcal{H}_1$ defined by $\overline{U}f = f\left(\tan\left(\frac{\theta}{2}\right)\right)\left(i+\tan\left(\frac{\theta}{2}\right)\right)\sqrt{\pi}$. By a similar calculation as above, we can show that $\left|\overline{U}f\right|_{\mathcal{H}_1} = |f|_{\mathcal{H}_2}$.

Let $f \in \mathcal{H}_2$ and $F \in \mathcal{H}_1$

Claim: $\overline{U} \circ U(F) = F$

Proof:

$$
\begin{aligned}
\overline{U} \circ U(F) &= \frac{1}{\sqrt{\pi}}\frac{1}{i+\tan\left(\frac{\theta}{2}\right)}F\left(\frac{i-\tan\left(\frac{\theta}{2}\right)}{i+\tan\left(\frac{\theta}{2}\right)}\right)\left(i+\tan\left(\frac{\theta}{2}\right)\right)\sqrt{\pi} \\
&= F\left(e^{i\theta}\right)
\end{aligned}
$$

Similarly, we can show that $U \circ \overline{U}(f) = f$ and hence $\overline{U} = U^{-1}$ and thus U must be onto.

Combining all of these, we see that U is a unitary correspondence between $\mathcal{H}_1$ and $\mathcal{H}_2$

We know that $\{\phi_n\}_{-\infty}^{\infty}$ where $\phi_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$ is an orthogonal basis of $\mathbb{L}^2[-\pi,\pi]$. This means that $F_n\left(e^{i\theta}\right) = e^{in\theta}$ is an orthonormal basis for $\mathcal{H}_1$. Then the claim is that $f_n(x) = UF_n = \frac{1}{\sqrt{\pi}}\left(\frac{i+x}{i-x}\right)^n \frac{1}{i+x}$ forms an orthonormal basis for $\mathbb{L}^2(\mathbf{R})$

$$
|F_n|_{\mathcal{H}_1} = |UF_n|_{\mathcal{H}_2} \quad (F_n, F_m)_{\mathcal{H}_1} = (UF_n, UF_m)_{\mathcal{H}_2}
$$

Hence $\{f_n\}$ is orthonormal in $\mathcal{H}_2$. Let $f \in \mathcal{H}_2$. Then $\overline{U}f \in \mathcal{H}_1$. Suppose f is orthogonal to all the basis vectors f_n then $(f, f_n)_{\mathcal{H}_2} = 0$ for all n.

$$(f, f_n)_{\mathcal{H}_2} = \left(\overline{U} f, F_n\right)_{\mathcal{H}_1} = 0$$

Hence $\overline{U} f \equiv 0$ since it is orthogonal to all basis elements F_n. $\therefore U \circ \overline{U} f \equiv 0$ and hence $f \equiv 0$

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6. Prove that the operator $T : \mathbb{L}^2[0, \infty] \to \mathbb{L}^2[0, \infty]$

$$T[f](x) = \frac{1}{\pi} \int_0^\infty \frac{f(y)}{x+y} \, dy$$

is bounded operator with norm $\|T\| \leq 1$.

Solution: Use problem 4, practice problem set 1, with $w(x) = \frac{1}{\sqrt{x}}$.

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7. Suppose that the multiplication operator $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is defined via $Ae_n = \alpha_n e_n$ where $\{e_i\}_{i=1}^\infty$ are the standard coordinate vectors and $\alpha_n \in \mathbb{R}$. Then show that A is bounded if and only if $\sup_n |\alpha_n| \leq M$.

Solution: $\sup_n |\alpha_n| \leq M \implies A$ is bounded.
If $f \in \mathcal{H}$, then $f = \sum_n (f, e_n) e_n$ and $Af = \sum_n \alpha_n (f, e_n) e_n$. By Parseval,

$$\|f\|^2 = \sum_{n=1}^\infty |(f, e_n)|^2 \,,$$

and

$$\|Af\|^2 = \sum_{n=1}^\infty |\alpha_n (f, e_n)|^2 \leq M^2 \sum_n \|(f, e_n)\|^2 = M^2 \|f\|^2 \,.$$

Thus, A is bounded.
A is bounded $\implies \sup_n \|\alpha_n\| < \infty$.
Suppose not. Then there exists a subsequence n_k, $k = 1, 2, \ldots$, such that $|\alpha_{n_k}| \geq k$. A being bounded implies that there exists an $M < \infty$ such that $\|Af\| \leq M \|f\|$ for any $f \in \mathcal{H}$. However $\|Ae_{n_k}\| \geq k \|e_{n_k}\|$ holds for any k which is a contradiction.

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8. Suppose that $\mathcal{K} : \mathbb{L}^2([0, 1]) \to \mathbb{L}^2([0, 1])$ is defined by

$$\mathcal{K}[f] = \int_0^1 k(x, y) f(y) \, dy \,,$$

where $k(x,y) \in \mathbb{L}^2([0,1]\times[0,1])$. Show that $\mathcal{K}$ is a bounded linear operator.

Solution:

$$
\begin{aligned}
|\mathcal{K}[f](x)| &= |\int_0^1 k(x,y)f(y)dy| \\
&\leq \int_0^1 |k(x,y)f(y)|dy \\
&\leq \sqrt{\left(\int_0^1 |k(x,y)|^2 dy\right)} \cdot \|f\|_{\mathbb{L}^2[0,1]} \quad \text{(Hölder inequality)} \\
\therefore \int_0^1 \|\mathcal{K}[f](x)\|^2 dx &\leq \int_0^1 \int_0^1 |k(x,y)|^2 dy dx \cdot \|f\|^2_{\mathbb{L}^2[0,1]} \\
\therefore \|\mathcal{K}[f]\|_{\mathbb{L}^2[0,1]} &\leq \sqrt{\left(\int_0^1 \int_0^1 |k(x,y)|^2 dy dx\right)} \cdot \|f\|_{\mathbb{L}^2[0,1]}
\end{aligned}
$$

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9. Give two examples of linear subspaces of $\mathbb{L}^2(\mathbb{R})$ which are not closed and find their closure.

   Solution: There are many options here, for example, C^k functions, i.e. functions which have k continuous derivatives, which are compactly supported are both linear subspaces and dense in $\mathbb{L}^2(\mathbb{R})$ for any k.

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10. Suppose that P_1 and P_2 are orthogonal projections onto subspaces S_1 and S_2. Show that P_2P_1 is an orthogonal projection if and only if P_1 and P_2 commute, i.e. $P_1P_2 = P_2P_1$ and in this case P_2P_1 projects onto $S_2 \cap S_1$. Give an example of two projection operators which do not commute.

    Solution: From exercise 1 in the practice problem set, it is clear that $P_j = P_j^*$ and $P_j^2 = P_j$ for $j = 1, 2$. Suppose that P_1P_2 is an orthogonal projection. Then $P_1P_2 = (P_1P_2)^* = P_2^*P_1^* = P_2P_1$.

    Now suppose that $P_1P_2 = P_2P_1$, then for all $f \in \mathcal{H}$

$$
\begin{aligned}
(P_1P_2f, f - P_1P_2f) &= (P_2f, P_1^*f - P_1^*P_1P_2f) \\
&= (P_2f, P_1f - P_1^2P_2f) \\
&= (P_2f, P_1f - P_1P_2f) \\
&= (P_2f, P_1f - P_2P_1f) \\
&= (P_2f, (I - P_2)P_1f) \\
&= 0 \quad (\text{since } P_2f \in S_2 \text{ and } (I-P_2)P_1f \in S_2^\perp)
\end{aligned}
$$

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11. Let $\mathcal{H} = \mathbb{L}^2(\mathbb{R})$. Let $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be the Fourier transform

$$\mathcal{F}[f](x) = \int_{-\infty}^{\infty} e^{i2\pi xy} f(y)\, dy\,.$$

Then it is well known that $\mathcal{F}$ is a unitary map with the inverse

$$\mathcal{F}^{-1}[f](x) = \int_{-\infty}^{\infty} e^{-i2\pi xy} f(y)\, dy\,.$$

Let $f * g$ denote the convolution operator

$$f * g(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\, dy$$

Further, it is also known that

$$\mathcal{F}[fg](x) = \mathcal{F}[f] * \mathcal{F}[g]\,,$$

and

$$\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g]\,.$$

(a) Let $\chi_A(x)$ denote the indicator function of the set A, i.e. $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise. Suppose $k_0 > 0$. Show that

$$\mathcal{F}[\chi_{[-k_0,k_0]}] = \frac{\sin(2\pi k_0 x)}{\pi x}$$

(b) Let $K(x) = \mathcal{F}[\chi_{[-k_0,k_0]}](x)$. Show that

$$\int_{-\infty}^{\infty} K(x-z)K(z-y)\, dz = K(x-y)\,.$$

(c) Let $\mathcal{K} : \mathbb{L}^2(\mathbb{R}) \to \mathbb{L}^2(\mathbb{R})$ denote the operator defined by

$$\mathcal{K}[f](x) = \int_{-\infty}^{\infty} K(x-y)f(y)\, dy\,.$$

Show that $\mathcal{K}$ is a bounded operator.

(d) Use part (b) to show that $\mathcal{K}$ is a projection operator in the following sense, $\mathcal{K}[\mathcal{K}[f]] = \mathcal{K}[f]$

(e) Let $\mathcal{H}_0 \subset \mathcal{H}$ denote the subspace defined by:

$$f \in \mathcal{H}_0 \quad \text{if} \quad \mathcal{F}[f](x) = 0 \quad \forall |x| > k_0\,.$$

Show that $\mathcal{H}_0$ is a closed linear subspace. $\mathcal{H}_0$ is the subspace of band-limited functions with band-limit k_0.

(f) Show that $\mathcal{K}$ is the projection operator onto $\mathcal{H}_0$.

Solution: a)

$$\mathcal{F}[\chi_{[-k_0,k_0]}] = \int_{-\infty}^{\infty} e^{i2\pi xy}\chi_{[-k_0,k_0]}(y)dy$$
$$= \int_{-k_0}^{k_0} e^{i2\pi xy}dy = \frac{\sin(2\pi k_0 x)}{\pi x}$$

b) A simple calculation shows that

$$(f,g) = (\mathcal{F}^{-1}[f], \mathcal{F}^{-1}g)\,,$$

$$\mathcal{F}^{-1}[K(x-z)](\xi) = e^{-2\pi i\xi x}\chi_{[-k_0,k_0]}(\xi)\,,$$

and

$$\mathcal{F}^{-1}[K(z-y)](\xi) = e^{-2\pi i\xi y}\chi_{[-k_0,k_0]}(\xi)\,,$$

Combining these three results, we get

$$\int_{-\infty}^{\infty} K(x-z)\cdot K(z-y) = \int_{-\infty}^{\infty} K(x-z)\cdot\overline{K(z-y)}dz \quad \text{K is real}$$
$$= \int_{-\infty}^{\infty} \mathcal{F}^{-1}[K(x-z)](\xi)\cdot\overline{\mathcal{F}^{-1}[K(z-y)](\xi)}d\xi$$
$$= \int_{-\infty}^{\infty} e^{-2\pi i\xi(x-y)}\chi_{[-k_0,k_0]}(\xi)d\xi$$
$$= K(x-y)$$

c) We first note that

$$\mathcal{F}[\mathcal{K}[f]] = \mathcal{F}[K * f] = \mathcal{F}[K]\cdot\mathcal{F}[f] = \chi_{-[k_0,k_0]}(x)\cdot\mathcal{F}[f](x)\,. \tag{1}$$

Since, the Fourier transform is an isometry, we have

$$\|\mathcal{K}[f]\|. = \|\mathcal{F}[\mathcal{K}[f]]\| = \|\chi_{-[k_0,k_0]}(x)\cdot\mathcal{F}[f](x)\| \le \|\mathcal{F}[f]\| = \|f\|\,.$$

Thus $\mathcal{K}$ is a bounded operator with $\|\mathcal{K}\| \le 1$.
d) Follows from part *b*.
e) Suppose $f_n \to f$ in $\mathbb{L}^2(\mathbb{R})$, then $\mathcal{F}[f_n] \to \mathcal{F}[f]$ in $\mathbb{L}^2(\mathbb{R})$. Since $\mathcal{F}[f_n](x) = 0$ for all n and all $|x| > k_0$, we conclude that $\mathcal{F}[f](x) = 0$ for almost every x such that $|x| > k_0$.
f) From equation 1, it follows that $\mathcal{K}[f] \in \mathcal{H}_0$ for all $f \in \mathcal{H}$. Moreover

$$(\mathcal{K}[f], f-\mathcal{K}[f]) = (\mathcal{F}[\mathcal{K}[f]], \mathcal{F}[f-\mathcal{K}[f]]) = \int_{-\infty}^{\infty} \chi_{-[k_0,k_0]}\mathcal{F}[f]\cdot(1-\chi_{[-k_0,k_0]})\mathcal{F}[f] = 0$$