1. If \( \{h_n\} \) is a sequence in a Hilbert space \( \mathcal{H} \) such that \( \sum_n \|h_n\| < \infty \), then show that \( h_n \) converges.

2. Suppose that \( E \) is a linear subspace of a Hilbert space \( \mathcal{H} \), then show that the closure of \( E \) is also a linear subspace.

3. Suppose that \( E \) is a subspace of a Hilbert space \( \mathcal{H} \), then show that \( (E^\perp)^\perp \) is the closure of the span of elements in \( E \), i.e.

\[
(E^\perp)^\perp = \left\{ \sum_{j=1}^{N} c_j f_j, \ f_j \in E \right\}
\]

4. Suppose that \( \mathcal{H} = \ell^2(\mathbb{N}) \).
   (a) Show that if \( \{a_n\} \in \mathcal{H} \), then the power series \( \sum_{n=1}^{\infty} a_n z^n \) has radius of convergence at least 1.
   (b) For \( \lambda < 1 \), show that \( L(\{a_n\}) := \sum_{n=1}^{\infty} a_n \lambda^n \) is a bounded linear functional.
   (c) Find the element \( h_0 \in \mathcal{H} \) such that \( L(h) = (h, h_0) \) and find \( \|L\| \).

5. Let \( \mathcal{H}_1 = L^2([-\pi, \pi]) \) be the Hilbert space of functions \( F(e^{i\theta}) \) on the unit circle with the inner product

\[
(F, G) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{i\theta})\overline{G(e^{i\theta})}d\theta.
\]

Let \( \mathcal{H}_2 \) be the space \( L^2(\mathbb{R}) \). Using the mapping

\[
x \rightarrow \frac{i-x}{i+x}
\]

of \( \mathbb{R} \) to the unit circle, show that:

a) The correspondence \( U : \mathcal{H}_1 \rightarrow \mathcal{H}_2 \) given by

\[
U[F] = \frac{1}{\pi^{1/2}(i+x)} F\left(\frac{i-x}{i+x}\right)
\]
is a unitary mapping.
b) As a result show that
\[
\left\{ \frac{1}{\pi^{1/2}(i + x)} \left( \frac{i - x}{i + x} \right)^n \right\}_{n=-\infty}^{\infty}
\]
is an orthonormal basis of \( \mathbb{L}^2(\mathbb{R}) \).

6. Prove that the operator \( T : \mathbb{L}^2[0, \infty] \to \mathbb{L}^2[0, \infty] \)
\[
T[f](x) = \frac{1}{\pi} \int_0^\infty \frac{f(y)}{x + y} \, dy
\]
is bounded operator with norm \( \|T\| \leq 1 \).

7. Suppose that the multiplication operator \( A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \) is defined via \( Ae_n = \alpha_n e_n \) where \( \{e_i\}_{i=1}^\infty \) are the standard coordinate vectors and \( \alpha_n \in \mathbb{R} \). Then show that \( A \) is bounded if and only if \( \sup_n |\alpha_n| \leq M \).

8. Suppose that \( \mathcal{K} : \mathbb{L}^2([0, 1]) \to \mathbb{L}^2([0, 1]) \) is defined by
\[
\mathcal{K}[f] = \int_0^1 k(x, y) f(y) \, dy,
\]
where \( k(x, y) \in \mathbb{L}^2([0, 1] \times [0, 1]) \). Show that \( \mathcal{K} \) is a bounded linear operator.

9. Give two examples of linear subspaces of \( \mathbb{L}^2(\mathbb{R}) \) which are not closed and find their closure.

10. Suppose that \( P_1 \) and \( P_2 \) are orthogonal projections onto subspaces \( S_1 \) and \( S_2 \). Show that \( P_2P_1 \) is an orthogonal projection if and only if \( P_1 \) and \( P_2 \) commute, i.e. \( P_1P_2 = P_2P_1 \) and in this case \( P_2P_1 \) projects onto \( S_2 \cap S_1 \). Give an example of two projection operators which do not commute.

11. Let \( \mathcal{H} = \mathbb{L}^2(\mathbb{R}) \). Let \( \mathcal{F} : \mathcal{H} \to \mathcal{H} \) be the Fourier transform
\[
\mathcal{F}[f](x) = \int_{-\infty}^{\infty} e^{2\pi i xy} f(y) \, dx.
\]
Then it is well known that \( \mathcal{F} \) is a unitary map with the inverse
\[
\mathcal{F}^{-1}[f](x) = \int_{-\infty}^{\infty} e^{-i2\pi xy} f(y) \, dx.
\]
Let \( f * g \) denote the convolution operator
\[
f * g(x) = \int_{-\infty}^{\infty} f(x - y)g(y) \, dy
\]
Further, it is also known that
\[
\mathcal{F}[fg](x) = \mathcal{F}[f] \ast \mathcal{F}[g],
\]
and
\[
\mathcal{F}[f \ast g] = \mathcal{F}[f] \cdot \mathcal{F}[g].
\]
(a) Let $\chi_A(x)$ denote the indicator function of the set $A$, i.e. $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise. Suppose $k_0 > 0$. Show that

$$F[\chi_{[-k_0,k_0]}] = \frac{\sin(2\pi k_0 x)}{\pi x}.$$ 

(b) Let $K(x) = F[\chi_{[-k_0,k_0]}](x)$. Show that

$$\int_{-\infty}^{\infty} K(x-z)K(z-y) \, dz = K(x-y).$$

(c) Let $\mathcal{K}: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the operator defined by

$$\mathcal{K}[f](x) = \int_{-\infty}^{\infty} K(x-y)f(y) \, dy.$$

Show that $\mathcal{K}$ is a bounded operator.

(d) Use part (b) to show that $\mathcal{K}$ is a projection operator in the following sense, $\mathcal{K}[\mathcal{K}[f]] = \mathcal{K}[f]$

(e) Let $\mathcal{H}_0 \subset \mathcal{H}$ denote the subspace defined by:

$$f \in \mathcal{H}_0 \quad \text{if} \quad F[f](x) = 0 \quad \forall |x| > k_0.$$

Show that $\mathcal{H}_0$ is a closed linear subspace. $\mathcal{H}_0$ is the subspace of band-limited functions with band-limit $k_0$.

(f) Show that $\mathcal{K}$ is the projection operator onto $\mathcal{H}_0$. 
