Fredholm Theory

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Roughly speaking, Fredholm theory consists of the study of operators of the form $I + A$ where A is compact. From this point on, we will also refer to $I + A$ as Fredholm operators. These are typically the operators for which results from linear algebra naturally extend to infinite dimensional spaces.

Just to recap, $T : \mathcal{H} \to \mathcal{H}$ is a compact operator if it is well-approximated by finite-rank operators, i.e., it is the norm limit of finite rank operators, i.e., there exists T_n where T_n are finite-rank, such that

$$\|T_n - T\| \to 0 \quad \text{as} \quad n \to \infty$$

An alternate definition of compact operators is that the image of the unit ball is pre-compact. This implies that if f_n is a bounded sequence and T is a compact operator, then there exists a subsequence f_{n_k} such that Tf_{n_k} converges.

Remark 1. *While these definitions are equivalent on Hilbert spaces, in certain Banach spaces, the results are in fact not equivalent and the standard definition of compact operators in that setup is the latter one, i.e., the image of unit ball is pre-compact.*

Here are a few examples of compact operators.

1. Finite-rank operators: Since their range is finite dimensional, and bounded sets in finite dimensions are precompact

2. Diagonal operators with eigenvalues decaying to 0, i.e.

$$Te_k = \lambda_k e_k \,,$$

   where $\lambda_k \to 0$ as $k \to \infty$.

3. Integral operators with a continuous kernel on compact sets:

$$T[f](x) = \int_0^1 K(x, y) f(y) dy$$

   where $K(x, y) : [0, 1] \times [0, 1] \to \mathbb{R}$ is continuous. Here $T : \mathbb{L}^2[0, 1] \to \mathbb{L}^2[0, 1]$ is compact

4. Integral operators with weakly-singular kernels (often encountered in solution of partial differential equations)

$$T[f](x) = \int_G K(x,y)f(y)dy \,,$$

where $K(x,y)$ is continuous for $x \neq y$ and in the vicinity of $x = y$, K satisfies

$$|K(x,y)| \leq \frac{1}{|x-y|^\alpha} \,,$$

with $\alpha < n$ when $G \subset \mathbb{R}^n$. Here, $T : \mathbb{L}^2[G] \to \mathbb{L}^2[G]$ is also compact.

In finite dimensions, for any linear operator A, we know that

$$\mathcal{N}(A) = \text{Ran}(T^*)^\perp \,, \quad \text{Ran}(A = \mathcal{N}(A^*)^\perp \,,$$
$$\mathcal{N}(A^*) = \text{Ran}(A)^\perp \,, \quad \text{Ran}(A^*) = \mathcal{N}(A)^\perp \,.$$

Moreover

$$\dim(\mathcal{N}(A)) = \dim(\mathcal{N}(A^*)) < \infty \,.$$

The results on the right give a complete description for solutions of linear systems $Ax = b$. It says that if $\mathcal{N}(A) = \{0\}$, then there exists a unique solution x to the problem $Ax = b$ or every b. If $\dim(\mathcal{N}(A)) = k > 0$, then there exists a k dimensional family of solutions x to the problem $Ax = b$ for each b in the range of the operator A, and whether a vector b is in the range of A or not can be determined by testing a finite number of conditions. If $y_1, y_2 \ldots y_k$ form a basis for $\mathcal{N}(A^*)$ and if b satisfies $(b, y_j) = 0$, $j = 1, 2, \ldots k$, then b is in the range of A.

In infinite dimensions, for all linear operators we have that

$$\mathcal{N}(A) = \text{Ran}(A^*)^\perp \quad \text{and} \mathcal{N}(A^*) = \text{Ran}(A)^\perp \,.$$

However, the range of operators need not be necessarily closed and hence

$$\mathcal{N}(A)^\perp = \left(\text{Ran}(A^*)^\perp\right)^\perp = \overline{\text{Ran}(A^*)} \,,$$

and similarly

$$\mathcal{N}(A^*)^\perp = \left(\text{Ran}(A)^\perp\right)^\perp = \overline{\text{Ran}(A)} \,,$$

Even if A is compact, it is not necessary that the operator have closed range. Consider the diagonal operator T defined by $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ as $Te_k = \frac{e_k}{k^2}$, where e_k are the standard coordinate vectors in $\ell^2(\mathbb{N})$. Then a simple calculation shows that $\text{Ran}(T)$ is dense in $\ell^2(\mathbb{N})$, but does not contain the whole space. The range is dense, since all finite linear combinations of e_k are contained in the range of T. However, the range does not contain the vector

$$\left(1, \frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n}, \ldots\right) \notin \text{Ran}(T)$$

For the rest of the section, we set $T = I + A$ where A is compact.

In the next theorem, we show that Fredholm operator of the form $T = I + A$ have finite dimensional null-spaces.

Theorem 2. *If $T = I + A$, where $A : \mathcal{H} \to \mathcal{H}$ is compact, then*

$$dim(\mathcal{N}(T)) < \infty$$

Proof. Suppose not. Then there exists a collection of orthogonal vectors ϕ_k, $k = 1, 2, \ldots$, of norm one, i.e. $(\phi_k, \phi_j) = 0$ if $j \neq k$ and $\|\phi_k\| = 1$, such that $T\phi_k = 0$. Since ϕ_k is a bounded collection of vectors and A is compact, there exists a subsequence ϕ_{n_k} such that

$$A\phi_{n_k} \to \psi \in \mathcal{H} \quad \text{as} \quad k \to \infty\,.$$

Then

$$\phi_{n_k} = T\phi_{n_k} - A\phi_{n_k} = -A\phi_{n_k}\,.$$

Thus,

$$\phi_{n_k} \to -\psi \quad \text{as} \quad k \to \infty\,,$$

which is a contradiction, since ϕ_{n_k} are orthogonal to each other. ■

In the following theorem, we show that Fredholm operators of the form $T = I + A$ have a closed range.

Theorem 3. *If $T = I + A$, where $A : \mathcal{H} \to \mathcal{H}$ is compact, then Ran(T) is closed*

Proof. Suppose $f_n \in \mathrm{Ran}(T)$, $n = 1, 2, \ldots$, and that $f_n \to f$ as $n \to \infty$. Since $f_n \in \mathrm{Ran}(T)$, $\exists g_n$ such that

$$Tg_n = f_n\,.$$

Since $\mathcal{N}(T)$ is closed, $\mathcal{H} = \mathcal{N}(T) \oplus \mathcal{N}(T)^\perp$. Let $g_n = \chi_n + \phi_n$, where $\chi_n \in \mathcal{N}(T)$ and $\phi_n \in \mathcal{N}(T)^\perp$. Thus,

$$f_n = Tg_n = T(\chi_n + \phi_n) = T\phi_n\,.$$

Claim: ϕ_n, $n = 1, 2, \ldots$, is a bounded sequence. Suppose not, then there exists a subsequence ϕ_{n_k} such that $\|\phi_{n_k}\| \geq k$. Let $\psi_k = \phi_{n_k}/\|\phi_{n_k}\|$ Then

$$T\psi_k = T\frac{\phi_{n_k}}{\|\phi_{n_k}\|} = \frac{f_{n_k}}{\|\phi_{n_k}\|}$$

Since $f_n \to f$, f_{n_k}, $k = 1, 2, \ldots$ is a bounded sequence. Thus,

$$\lim_{k\to\infty} T\psi_k = \lim_{k\to\infty} \frac{f_{n_k}}{\|\phi_{n_k}\|} = 0$$

Furthermore, since ψ_k is a bounded sequence and A is compact, there exists a further subsequence ψ_{n_k} such that $A\psi_{n_{k,1}} \to \psi$. Moreover,

$$\lim_{k\to\infty} \psi_{n_{k,1}} = \lim_{k\to\infty} \left(T\psi_{n_{k,1}} - A\psi_{n_{k,1}}\right) = -\psi\,.$$

By the continuity of T, we further conclude that

$$T\psi = \lim_{k\to\infty} T\psi_{n_{k,1}} = 0\,.$$

Thus, $\psi \in \mathcal{N}(T)$. Then,

$$\begin{aligned}
\|\psi_{n_{k,1}} + \psi\|^2 &= \left\| \frac{\phi_{n_{n_{k,1}}}}{\|\phi_{n_{n_{k,1}}}\|} + \psi \right\|^2 \\
&= \frac{1}{\|\phi_{n_{n_{k,1}}}\|^2} \left\| \phi_{n_{n_{k,1}}} + \|\phi_{n_{n_{k,1}}}\| \psi \right\|^2 \\
&= \frac{1}{\|\phi_{n_{n_{k,1}}}\|^2} \left(\|\phi_{n_{n_{k,1}}}\|^2 (1 + \|\psi^2\|) \right) \quad (\text{since} \quad \psi \perp \phi_n) \\
&\geq 1 \,,
\end{aligned}$$

which contradicts $\psi_{n_{k,1}} \to -\psi$. This proves the claim.

So what we have now is that

$$T\phi_n = f_n \,,$$

where ϕ_n is a bounded sequence, and $f_n \to f$. Since ϕ_n is a bounded sequence, and A is compact, there exists a subsequence ϕ_{n_k} such that $A\phi_{n_k} \to g$ as $k \to \infty$. Then,

$$\lim_{k\to\infty} \phi_{n_k} = \lim_{k\to\infty} T\phi_{n_k} - A\phi_{n_k} = \lim_{k\to\infty} f_{n_k} - A\phi_{n_k} = f - g$$

By the continuity of T then,

$$T(f-g) = \lim_{k\to\infty} T\phi_{n_k} = \lim_{k\to\infty} f_{n_k} = f$$

Thus, $f \in \mathrm{Ran}(T)$ and we conclude that $\mathrm{Ran}\,(T)$ is closed. ■

At this stage, we conclude that

$$\mathcal{N}(T)^\perp = \left(\mathrm{Ran}(T^*)^\perp\right)^\perp = \overline{\mathrm{Ran}(T^*)} = \mathrm{Ran}\,(T^*) \,,$$

and that

$$\mathcal{N}(T^*)^\perp = \left(\mathrm{Ran}(T)^\perp\right)^\perp = \overline{\mathrm{Ran}(T)} = \mathrm{Ran}\,(T) \,,$$

for all Fredholm operators of the form $T = I + A$.

Remark 4. *In many references, compact operators are also referred to as first kind Fredholm operators, and $T = I + A$, where A is compact are referred to as second kind Fredholm operators.*

We note that if $T = I + A$, then $T^n = I + A_n$ where A_n is also compact, since

$$T^n = (I + A)^n = \sum_{k=0}^{n} \binom{n}{k} A^k = I + \sum_{k=1}^{n} \binom{n}{k} A^k$$

since each of A^k, $k = 1, 2, \ldots n$ are compact, we conclude that $T = I + A_n$.

We note that if $\phi \in \mathcal{N}(T^k)$, then $\phi \in \mathcal{N}(T^{k+1})$ for any k, since

$$T^k\phi = 0 \implies T^{k+1}\phi = T(T^k\phi) = 0 \,.$$

Thus

$$\{0\} = \mathcal{N}(T^0) \subseteq \mathcal{N}(T) \subseteq \mathcal{N}(T^2) \ldots \subseteq \mathcal{N}(T^k) \subseteq \mathcal{N}(T^{k+1}) \ldots$$

Similarly, if $\phi \in \text{Ran}\left(T^k\right)$, then $\phi \in \text{Ran}\left(T^{k-1}\right)$ for all $k \geq 1$, since

$$\phi \in \text{Ran}\left(T^k\right) \implies \exists \psi \quad \text{such that} \quad \phi = T^k\psi = T^{k-1}(T\psi) \implies \phi \in \text{Ran}\left(T^{k-1}\right) \,.$$

Thus,

$$\mathcal{H} = \text{Ran}\left(T^0\right) \supseteq \text{Ran}\left(T\right) \supseteq \text{Ran}\left(T^2\right) \ldots \supseteq \text{Ran}\left(T^k\right) \supseteq \text{Ran}\left(T^{k+1}\right) \ldots$$

Another feature of Fredholm operators of the form $T = I + A$ is that the null spaces and ranges cannot go nesting indefinitely. The nesting stops at some point. In fact, the point at which the nesting stops is the same, and the corresponding range and null space form a direct decomposition of the whole space. Such a result is not necessarily true for compact operators. For example, consider the right shift operator scaled appropriately.

Remark 5. *A combination of right/left shift operators with diagonal scalings are often a good place to start when hunting for counter-examples.*

We prove the result for Fredholm operators of the form $T = I + A$ in the following theorem which is also referred to as Reisz's theorem.

Theorem 6. *There exists* $0 < r < \infty$ *such that*

$$\{0\} = \mathcal{N}(T^0) \subsetneq \mathcal{N}(T) \subsetneq \mathcal{N}(T^2) \ldots \mathcal{N}(T^{r-1}) \subsetneq \mathcal{N}(T^r) = \mathcal{N}(T^{r+1}) = \mathcal{N}(T^{r+2}) = \ldots \,,$$

and

$$\mathcal{H} = Ran\left(T^0\right) \supsetneq Ran\left(T\right) \supsetneq Ran\left(T^2\right) \ldots Ran\left(T^{r-1}\right) \supsetneq Ran\left(T^r\right) = Ran\left(T^{r+1}\right) = Ran\left(T^{r+2}\right) \ldots$$

Furthermore, $\mathcal{H} = \mathcal{N}(T^r) \oplus Ran\left(T^r\right)$.

Proof. We will prove the results in four parts.

1. **Claim**: There exists $0 < p < \infty$ such that

$$\{0\} = \mathcal{N}(T^0) \subsetneq \mathcal{N}(T) \subsetneq \mathcal{N}(T^2) \ldots \mathcal{N}(T^{r-1}) \subsetneq \mathcal{N}(T^p) = \mathcal{N}(T^{p+1}) = \mathcal{N}(T^{p+2}) = \ldots \,, \tag{1}$$

Suppose not. Suppose that

$$\{0\} = \mathcal{N}(T^0) \subsetneq \mathcal{N}(T) \subsetneq \mathcal{N}(T^2) \ldots \mathcal{N}(T^{r-1}) \subsetneq \mathcal{N}(T^p) \subsetneq \mathcal{N}(T^{p+1}) \subsetneq \ldots$$

Then, there exists $\phi_k \in \mathcal{N}(T^{k+1})$, $\phi_k \perp \mathcal{N}(T^k)$ and $\|\phi_k\| = 1$. If $n > m$, then

$$\|A\phi_n - A\phi_m\|^2 = \|T\phi_n - T\phi_m - (\phi_n - \phi_m)\|^2$$

We note that $T\phi_n - T\phi_m + \phi_m \in \mathcal{N}(T^n)$. Thus $\phi_n \perp T\phi_n - T\phi_m + \phi_m$, and

$$\|A\phi_n - A\phi_m\|^2 = \|\phi_n\|^2 + \|T\phi_n - T\phi_m + \phi_m\|^2 \geq \|\phi_n\|^2 = 1 \,.$$

Thus, $A\phi_n$ cannot have a convergent subsequence, which contradicts the compactness of A

Therefore, there exists some p such that

$$\mathcal{N}(T^p) = \mathcal{N}(T^{p+1}) .$$

Subclaim: for any $q \in \mathbb{N}$

$$\mathcal{N}(T^q) = \mathcal{N}(T^{q+1}) \implies \mathcal{N}(T^{q+1}) = \mathcal{N}(T^{q+2})$$

It is clear that

$$\mathcal{N}(T^{q+1}) \subset \mathcal{N}(T^{q+2}) .$$

Suppose that $\phi \in \mathcal{N}(T^{q+2})$, then $T\phi \in \mathcal{N}(T^{q+1})$. Since $\mathcal{N}(T^{q+1}) = \mathcal{N}(T^q)$, we conclude that $T\phi \in \mathcal{N}(T^q)$, and thus $\phi \in \mathcal{N}(T^{q+1})$. This proves the subclaim and also the result (1).

2. **Claim**: There exists $0 < r < \infty$ such that

$$\mathcal{H} = \operatorname{Ran}\left(T^0\right) \supsetneq \operatorname{Ran}(T) \supsetneq \operatorname{Ran}\left(T^2\right) \ldots \operatorname{Ran}\left(T^{r-1}\right) \supsetneq \operatorname{Ran}(T^r) = \operatorname{Ran}\left(T^{r+1}\right) = \operatorname{Ran}\left(T^{r+2}\right) \ldots \tag{2}$$

Suppose not. Suppose that

$$\mathcal{H} = \operatorname{Ran}\left(T^0\right) \supsetneq \operatorname{Ran}(T) \supsetneq \operatorname{Ran}\left(T^2\right) \ldots \operatorname{Ran}\left(T^{r-1}\right) \supsetneq \operatorname{Ran}(T^r) \supsetneq \operatorname{Ran}\left(T^{r+1}\right) \ldots$$

Then, there exists $\phi_k \in \operatorname{Ran}\left(T^{k-1}\right)$, $\phi_k \perp \operatorname{Ran}\left(T^k\right)$ and $\|\phi_k\| = 1$. If $n > m$, then

$$\|A\phi_n - A\phi_m\|^2 = \|T\phi_n - T\phi_m - (\phi_n - \phi_m)\|^2$$

We note that $T\phi_n - T\phi_m - \phi_n \in \operatorname{Ran}(T^m)$. Thus $\phi_m \perp T\phi_n - T\phi_m - \phi_n$, and

$$\|A\phi_n - A\phi_m\|^2 = \|\phi_m\|^2 + \|T\phi_n - T\phi_m - \phi_n\|^2 \geq \|\phi_m\|^2 = 1 .$$

Thus, $A\phi_n$ cannot have a convergent subsequence, which contradicts the compactness of A

Therefore, there exists some r such that

$$\operatorname{Ran}(T^r) = \operatorname{Ran}\left(T^{r+1}\right) .$$

Subclaim: for any $q \in \mathbb{N}$

$$\operatorname{Ran}(T^q) = \operatorname{Ran}\left(T^{q+1}\right) \implies \operatorname{Ran}\left(T^{q+1}\right) = \operatorname{Ran}\left(T^{q+2}\right)$$

It is clear that

$$\operatorname{Ran}\left(T^{q+1}\right) \supset \operatorname{Ran}\left(T^{q+2}\right) .$$

Suppose that $\phi \in \operatorname{Ran}(T^{q+1})$, then $\exists$ g such that $\phi = T^{q+1}g = TT^q g$. Since $\operatorname{Ran}(T^{q+1}) = \operatorname{Ran}(T^q)$, and $T^q g \in \operatorname{Ran}(T^q)$, we conclude that $T^q g \in \operatorname{Ran}(T^{q+1})$, i.e. $\exists$ h, such that, $T^q g = T^{q+1}h$. Thus,

$$\phi = T \cdot T^q g = T \cdot T^{q+1} h = T^{q+2} h ,$$

which shows that $\phi \in \operatorname{Ran}(T^{q+2})$. This proves the subclaim and also the result (2).

3. **Claim**: $p = r$. Suppose not. Suppose first that $p > r$. In this case, we will show that p was not minimal i.e. the condition that

$$\mathcal{N}(T^{p-1}) \subsetneq \mathcal{N}(T^p)\,,$$

is violated. Let $\phi \in \mathcal{N}(T^p)$. Then $T^p\phi = T \cdot T^{p-1}\phi = 0$. Since $p > r$, $\text{Ran}\,(T^{p-1}) = \text{Ran}\,(T^p)$. Thus, $\exists g$ such that $T^{p-1}\phi = T^p g$. Then $0 = T \cdot T^{p-1}\phi = T \cdot T^p g = T^{p+1}g$. Thus, $g \in \mathcal{N}(T^{p+1})$. However, $\mathcal{N}(T^{p+1}) = \mathcal{N}(T^p)$, thus $g \in \mathcal{N}(T^p)$, which implies that

$$T^{p-1}\phi = T^p g = 0\,,$$

Thus, $\phi \in \mathcal{N}(T^{p-1})$ and we conclude that $\phi \in \mathcal{N}(T^p) \implies \phi \in \mathcal{N}(T^{p-1})$. Recall that $\mathcal{N}(T^{p-1}) \subset \mathcal{N}(T^p)$ and thus we conclude that $\mathcal{N}(T^{p-1}) = \mathcal{N}(T^p)$ which is a contradiction.

Suppose now that $p < r$. In this case, we will show that r was not minimal, i.e. the condition that

$$\text{Ran}\left(T^{r-1}\right) \supsetneq \text{Ran}\,(T^r)\ ,$$

is violated. Suppose that $\phi \in \text{Ran}\,(T^{r-1})$, i.e. $\exists f$ such that $\phi = T^{r-1}f$ Then $T\phi \in \text{Ran}\,(T^r)$. Since $\text{Ran}\,(T^r) = \text{Ran}\,(T^{r+1})$, we conclude that $\exists g$ such that

$$T\phi = T^r f = T^{r+1}g\,.$$

Thus, $T^r(f - Tg) = 0$ and $f - Tg \in \mathcal{N}(T^r)$. Since $p > r$, $\mathcal{N}(T^r) = \mathcal{N}(T^{r-1})$, from which it follows that

$$f - Tg \in \mathcal{N}(T^{r-1}) \implies T^{r-1}(f - Tg) = 0 \implies \phi = T^{r-1}f = T^r g\,.$$

Thus, $\phi \in \text{Ran}\,(T^{r-1}) \implies \phi \in \text{Ran}\,(T^r)$. This combined with $\text{Ran}\,(T^r) \subset \text{Ran}\,(T^{r-1})$ contradicts the minimality of r.

4. Finally, we now show that $\mathcal{H} = \mathcal{N}(T^r) \oplus \text{Ran}\,(T^r)$. We first show that if such a decomposition exists, it must be unique. Suppose $f \in \mathcal{N}(T^r) \cap \text{Ran}\,(T^r)$. Then $T^r f = 0$, and there exists g such that $f = T^r g$. Since $T^r f = 0$, we note that $T^{2r}g = T^r f = 0$, i.e. $g \in \mathcal{N}(T^{2r})$. However, $\mathcal{N}(T^{2r}) = \mathcal{N}(T^r)$ from which we conclude that $T^r g = 0$ and thus $f = T^r g = 0$.

   As for existence, let $\phi \in \mathcal{H}$. Then $T^r\phi \in \text{Ran}\,(T^r)$. Moreover, since $\text{Ran}\,(T^r) = \text{Ran}\,(T^{2r})$, there exists f such that

$$T^r\phi = T^{2r}f\,.$$

   Thus, $\phi - T^r f \in \mathcal{N}(T^r)$, i.e. $\phi = T^r f + g$ where $g \in \mathcal{N}(T^r)$, and clearly $T^r f \in \text{Ran}\,(T^r)$.

■

A key result for Fredholm operators is that injectivity implies surjectivity + bounded inverse, which we prove in the theorem below.

Theorem 7. *Suppose that $T = I + A$ where A is compact. If T is injective, then is surjective and has a bounded inverse.*

Proof. Since T is injective, $\mathcal{N}(T) = \{0\}$. Thus, $\mathcal{N}(T^0) = \mathcal{N}(T^1)$ and thus the Reisz index of the operator T, denoted by r in the previous theorem is $r = 1$. Thus, $\mathcal{H} = \mathcal{N}(T) \oplus \text{Ran}\,(T)$. Since $\mathcal{N}(T) = \{0\}$, we conclude that $\text{Ran}\,(T) = \mathcal{H}$ and thus T is surjective.

We prove that T has a bounded inverse by contradiction. Suppose T^{-1} is not bounded, then there exists a sequence f_n, such that $\|f_n\| = 1$ and $\|T^{-1} f_n\| \geq n$. Let $g_n = T^{-1} f_n$, then $\|g_n\| \geq n$. $f_n = T g_n$ and

$$\frac{f_n}{\|g_n\|} = T\left(\frac{g_n}{\|g_n\|}\right)$$

Since $\|f_n\| = 1$ and $\|g_n\| \geq n$, we conclude that

$$\lim_{n\to\infty} \frac{f_n}{\|g_n\|} = 0\,.$$

Moroever, since $g_n/\|g_n\|$ is a bounded sequence and A is compact, there exists a convergent subseqeunce

$$\lim_{k\to\infty} A\frac{g_{n_k}}{\|g_{n_k}\|} = \psi\,,$$

Then

$$\lim_{k\to\infty} \frac{g_{n_k}}{\|g_{n_k}\|} = \lim_{k\to\infty} T\frac{g_{n_k}}{\|g_{n_k}\|} - A\frac{g_{n_k}}{\|g_{n_k}\|} = \lim_{k\to\infty} \frac{f_{n_k}}{\|g_{n_k}\|} - A\frac{g_{n_k}}{\|g_{n_k}\|} = 0 - \psi$$

Moreover, by the continuity of T

$$T\psi = \lim_{k\to\infty} T\frac{g_{n_k}}{\|g_{n_k}\|} = \lim_{k\to\infty} \frac{f_{n_k}}{\|g_{n_k}\|} = 0\,,$$

Since T is injective, we conclude that $\psi = 0$ which is a contradiction since a subsequence of norm 1 vectors $(g_{n_k}/\|g_{n_k}\|)$ is converging to 0. ■

Another special feature of Fredholm operators of the form $T = I + A$ is that the dimension of the null spaces of T and T^* are the same.

Theorem 8. *If $T = I + A$ where A is compact, then*

$$dim(\mathcal{N}(T)) = dim(\mathcal{N}(T^*))\,.$$

Proof. Suppose that $m = \dim(\mathcal{N}(T))$ and $n = \dim(\mathcal{N}(T^*))$.

Case 1: Suppose that $m = 0$. Then $\dim(\mathcal{N}(T)) = 0$. From theorem 6, we note that $\text{Ran}\,(T) = \mathcal{H}$. Moreover, $\mathcal{N}(T^*) = \text{Ran}\,(T)^{\perp}$, thus, we conclude that $\mathcal{N}(T^*) = \{0\}$ and that $n = 0$.

Case 2: Suppose that $n = 0$. A similar argument shows that $m = 0$.

Case 3: $m, n > 0$. In this case, we will now construct a Fredholm operator of the form $T_1 = I + A_1$ where A_1 is compact, which has the following properties. $\dim(\mathcal{N}(T_1) = m - 1$, and $\dim(\mathcal{N}(T_1^*)) = n - 1$. If we can construct such an operator, then using a recursive argument, we can construct an operator T_n if $m > n$ such that $\dim\mathcal{N}(T_n^*) = 0$, in which case we can appeal to case 2, or an operator T_m if $m < n$ such that $\dim\mathcal{N}(T_m^*) = 0$, in which case we can appeal to case 1 and we are done.

Construction of T_1:

Suppose that $f_1, f_2 \dots f_m$ is an orthogonal basis for $\mathcal{N}(T)$ and $g_1, g_2, \dots g_n$ is an orthogonal basis for $\mathcal{N}(T^*)$. Then define

$$T_1 = T - (\cdot, f_1)g_1 \,,$$

i.e.

$$T_1 h = Th - (h, f_1)g_1 \quad \forall h \in \mathcal{H}$$

Suppose that $h \in \mathcal{N}(T_1)$. Then,

$$T_1 h = Th - (h, f_1)g_1 = 0 \implies Th = (h, f_1)g_1$$

Since $g_1 \in \mathcal{N}(T^*) = \text{Ran}\,(T)^\perp$ and $Th \in \text{Ran}\,(T)$, we conclude that $Th = 0$ and $(h, f_1) = 0$. Since $f_1, f_2, \dots f_m$ form a basis for $\mathcal{N}(T)$, there exists $\alpha_j \in \mathcal{F}$ such that $h = \sum_{j=1}^m \alpha_j f_j$. However, $(h, f_1) = 0 \implies \alpha_1 = 0$ and thus, $h = \sum_{j=2}^m \alpha_j f_j$.

Conversely, if $h = \sum_{j=2}^m \alpha_j f_j$, then $Th = 0$ (since $h \in \mathcal{N}(T)$) and $(h, f_1) = 0$ (since $f_j \perp f_1$ for $j \neq 1$). Thus if $h = \sum_{j=2}^m \alpha_j f_j$, then $h \in \mathcal{N}(T_1)$.

These two results imply that $h \in \mathcal{N}(T_1)$ if and only if

$$h = \sum_{j=2}^m \alpha_j f_j \,,$$

which shows that $\dim(\mathcal{N}(T_1)) = m - 1$.

The adjoint of T_1 is given by

$$T_1^* = T^* + (\cdot, g_1)f_1 \,,$$

and a similar calculation shows that $\dim(\mathcal{N}(T^*)) = n - 1$ which concludes the proof. ■

Remark 9. *Fredholm operators are more generally defined to be operators which have a finite dimensional null space, whose range is closed and the adjoints also have a finite dimensional null space. In general, for Fredholm operators as well, the null spaces of T and T^* need not be the same. The difference between the dimension of the null space of the operator and its adjoint is referred to as the index of the operator. Thus, the statement above can be restated as, if T is a Fredholm operator of the form $I + A$, then it has index 0. Classical examples of Fredholm operators with non-zero index are the left and right shift operator (Verify this).*

1 Extension of results to the case of Banach spaces

For the rest of this section, we will assume that X is a Banach space and that X^* is the dual of X. A key property that was used over and over again in the derivation of the above results was that given $U \subset V$ where V is a closed subspace of X and U is a closed linear subspace of V which satisfies $V \setminus U \neq \emptyset$, then there exists a $\phi \in V \setminus U$ such that $\|\phi\| = 1$ and that $\phi \perp U$. An alternate formulation of the condition $\phi \perp U$ is that for all g in U,

$$\|\phi - g\|^2 = \|\phi\|^2 - 2(\phi, g) + \|g\|^2 \geq 1 \, .$$

This also implies that there exists a ϕ with $\|\phi\| = 1$ such that

$$d(\phi, U) = \inf_{g \in U} \|\phi - g\| = 1$$

In Banach spaces, the above result does not hold in general. However, a slightly weaker version of the above result is still true and is stated below.

Lemma 10. *Suppose that V is a closed subspace of X and that U is a closed linear subspace of V such that $V \setminus U \neq \emptyset$. Then, for any $\alpha < 1$, there exists $f \in V$ with $\|f\| = 1$ such that*

$$d(f, U) = \inf_{g \in U} \|f - g\| \geq \alpha \, .$$

Proof. Let $\psi \in V \setminus U$. Since U is a closed linear subspace of V, it follows that

$$d(\psi, U) = \inf_{g \in U} \|\psi - g\| = \beta > 0 \, .$$

Note that β here cannot be zero. Moreover, since $\alpha < 1$, there exists a $\psi_0 \in U$ such that

$$\beta \leq \|\psi - \psi_0\| \leq \frac{\beta}{\alpha} \, . \tag{3}$$

Then, f defined via

$$f = \frac{\psi - \psi_0}{\|\psi - \psi_0\|} \, ,$$

satisfies the desired inequality. For any $g \in U$

$$\begin{aligned} \|f - g\| &= \left\| \frac{\psi - \psi_0}{\|\psi - \psi_0\|} - g \right\| \\ &= \frac{1}{\|\psi - \psi_0\|} \|\psi - (g\|\psi - \psi_0\| + \psi_0)\| \, . \end{aligned}$$

Since $g, \psi_0 \in U$ and U is a linear subspace, $g\|\psi - \psi_0\| + \psi_0 \in U$. By definition, $d(\psi, U) \geq \beta$ which in particular implies that $\|\psi - (g\|\psi - \psi_0\| + \psi_0)\| \geq \beta$. Moreover, from equation 3, it follows that

$$\frac{1}{\|\psi - \psi_0\|} \geq \frac{\alpha}{\beta} \, .$$

Combining these estimates, we observe that

$$\|f - g\| = \frac{1}{\|\psi - \psi_0\|} \|\psi - (g\|\psi - \psi_0\| + \psi_0)\| \geq \frac{\alpha}{\beta} \cdot \beta = \alpha \, .$$

■

In the next theorem, we show that Fredholm operator of the form $T = I + A$ have finite dimensional null-spaces.

Theorem 11. *If $T = I + A$, where $A : X \to X$ is compact, then*

$$dim(\mathcal{N}(T)) < \infty$$

Proof. Suppose not. Then there exists a collection of vectors ϕ_k, $k = 1, 2, \ldots$, of norm one which satisfy

$$\|\phi_k - \phi_j\| \geq \frac{1}{2}\,.$$

To see, that since the null space is infinite dimensional, there exists a sequence of linearly independent vectors $x_1, x_2, \ldots x_k, \ldots$, which satisfy $Tx_i = 0$ for all i. Let $E_k = \text{span}(x_1, x_2, \ldots x_k)$. Then each E_k is a closed linear subspace, with $E_k \subset E_{k+1}$, $E_{k+1} \setminus E_k \neq \emptyset$, and $Tf = 0$ for all $f \in E_k$ for all k. Set $\phi_1 = x_1/\|x_1\|$. By the above lemma, we know that there exists $\phi_k \in E_{k+1}$ which satisfies $\|\phi_k\| = 1$, and $d(\phi_k, E_k) \geq \frac{1}{2}$. In particular $\|\phi_k - \phi_j\| \geq \frac{1}{2}$ for all $j = 1, 2, \ldots k-1$, since $\phi_j \in E_k$ for all $j = 1, 2, \ldots k-1$. Moreover, since $\phi_k \in E_{k+1}$, we note that $T\phi_k = 0$. This is our desired sequence of vectors ϕ. Since ϕ_k is a bounded collection of vectors and A is compact, there exists a subsequence ϕ_{n_k} such that

$$A\phi_{n_k} \to \psi \in \mathcal{H} \quad \text{as} \quad k \to \infty\,.$$

Then

$$\phi_{n_k} = T\phi_{n_k} - A\phi_{n_k} = -A\phi_{n_k}\,.$$

Thus,

$$\phi_{n_k} \to -\psi \quad \text{as} \quad k \to \infty\,,$$

which is a contradiction, since ϕ_{n_k} are at least a distance 1/2 from each other. ■

In the following theorem, we show that Fredholm operators of the form $T = I + A$ have a closed range.

Theorem 12. *If $T = I + A$, where $A : \mathcal{H} \to \mathcal{H}$ is compact, then Ran(T) is closed*

Proof. Suppose $f_n \in \text{Ran}(T)$, $n = 1, 2, \ldots$, and that $f_n \to f$ as $n \to \infty$. Since $f_n \in \text{Ran}(T)$, $\exists g_n$ such that

$$Tg_n = f_n\,.$$

A critical step in the proof in the case of Hilbert space was to show that there exists χ_n such that $T\phi_n = Tg_n = f_n$ where $\phi_n = P_{\mathcal{N}(T)^\perp} g_n$, and ϕ_n is a bounded sequence. Here $P_{\mathcal{N}(T)^\perp}$ denotes the orthogonal projection onto the subspace $\mathcal{N}(T)^\perp$.

Since, we do not have projection operators in Banach spaces, we will show that For each g_n, we find $\chi_n \in \mathcal{N}(T)$ which is the best approximation of g_n in $\mathcal{N}(T)$, i.e.

$$\|g_n - \chi_n\| = \inf_{\chi} \|g_n - \chi\|\,,$$

and show that $\phi_n = g_n - \chi_n$ is a bounded sequence. Firstly, there exists a closest element χ_n in $\mathcal{N}(T)$, even if X is not reflexive (note that we've shown in homework assignments, that there exists a unique closest element to a closed linear subspace in reflexive Banach spaces). The reason for this is that $\mathcal{N}(T)$ is a finite dimensional closed linear subspace.

Claim: ϕ_n, $n = 1, 2, \ldots$, is a bounded sequence. Suppose not, then there exists a subsequence ϕ_{n_k} such that $\|\phi_{n_k}\| \geq k$. Let $\psi_k = \phi_{n_k}/\|\phi_{n_k}\|$ Then

$$T\psi_k = T\frac{\phi_{n_k}}{\|\phi_{n_k}\|} = \frac{f_{n_k}}{\|\phi_{n_k}\|}$$

Since $f_n \to f$, f_{n_k}, $k = 1, 2, \ldots$ is a bounded sequence. Thus,

$$\lim_{k\to\infty} T\psi_k = \lim_{k\to\infty} \frac{f_{n_k}}{\|\phi_{n_k}\|} = 0$$

Furthermore, since ψ_k is a bounded sequence and A is compact, there exists a further subsequence ψ_{n_k} such that $A\psi_{n_{k,1}} \to \psi$. Moreover,

$$\lim_{k\to\infty} \psi_{n_{k,1}} = \lim_{k\to\infty} \left(T\psi_{n_{k,1}} - A\psi_{n_{k,1}}\right) = -\psi\,.$$

By the continuity of T, we further conclude that

$$T\psi = \lim_{k\to\infty} T\psi_{n_{k,1}} = 0\,.$$

Thus, $\psi \in \mathcal{N}(T)$. Then,

$$\begin{aligned}
\|\psi_{n_{k,1}} + \psi\| &= \left\| \frac{\phi_{n_{n_{k,1}}}}{\|\phi_{n_{n_{k,1}}}\|} + \psi \right\| \\
&= \frac{1}{\|\phi_{n_{n_{k,1}}}\|} \left\| \phi_{n_{n_{k,1}}} + \|\phi_{n_{n_{k,1}}}\|\psi \right\| \\
&= \frac{1}{\|\phi_{n_{n_{k,1}}}\|} \left\| g_{n_{n_{k,1}}} - \chi_{n_{n_{k,1}}} + \|\phi_{n_{n_{k,1}}}\|\psi \right\| \quad (\phi = g - \chi) \\
&= \frac{1}{\|\phi_{n_{n_{k,1}}}\|} \inf_{\chi\in\mathcal{N}(T)} \|g_{n_{n_{k,1}}} - \chi\| \quad (\text{since} \quad \psi\|\phi\| + \chi \in \mathcal{N}(T)) \\
&= \frac{1}{\|\phi_{n_{n_{k,1}}}\|} \|g_{n_{n_{k,1}}} - \chi_{n_{n_{k,1}}}\| = \frac{1}{\|\phi_{n_{n_{k,1}}}\|} \cdot \|\phi_{n_{n_{k,1}}}\| = 1
\end{aligned}$$

which contradicts $\psi_{n_{k,1}} \to -\psi$. This proves the claim.

So what we have now is that

$$T\phi_n = f_n\,,$$

where ϕ_n is a bounded sequence, and $f_n \to f$. Since ϕ_n is a bounded sequence, and A is compact, there exists a subsequence ϕ_{n_k} such that $A\phi_{n_k} \to g$ as $k \to \infty$. Then,

$$\lim_{k\to\infty} \phi_{n_k} = \lim_{k\to\infty} T\phi_{n_k} - A\phi_{n_k} = \lim_{k\to\infty} f_{n_k} - A\phi_{n_k} = f - g$$

By the continuity of T then,

$$T(f-g) = \lim_{k\to\infty} T\phi_{n_k} = \lim_{k\to\infty} f_{n_k} = f$$

Thus, $f \in \text{Ran}(T)$ and we conclude that $\text{Ran}\,(T)$ is closed. ■

We remark a few things at this stage.

- If $T : X \to X$ is a bounded linear operator, then $T^* : X^* \to X^*$ is also a bounded linear operator.

- For any subspace $U \subset X$, we define $U^\perp \subset X^*$ as

$$f \in U^\perp \subset X^* \quad \text{if} \quad f(x) = 0 \quad \forall x \in U\,.$$

- Similarly, for any subspace $V \subset X^*$, we define ${}^\perp V \subset X$ as

$$x \in^\perp V \subset X \quad \text{if} \quad f(x) = 0 \quad \forall f \in V\,.$$

- If U is a linear subspace of X, then ${}^\perp U^\perp = \overline{U}$

- If V is a linear subspace of X^*, then ${}^\perp V^\perp = \overline{V}$

- From these definitions, for any bounded operator T, we can show that

$$\mathcal{N}(T) =^\perp \text{Ran}(T^*) \quad \text{and} \quad \mathcal{N}(T^*) = \text{Ran}(T)^\perp$$

- For second kind Fredholm operators T, we get to specialize the result further, and conclude that

$$\mathcal{N}(T)^\perp = \left({}^\perp\text{Ran}(T^*)\right)^\perp = \overline{\text{Ran}(T^*)} = \text{Ran}\left(T^*\right)\,,$$

and that

$${}^\perp\mathcal{N}(T^*) =^\perp \left(\text{Ran}(T)^\perp\right) = \overline{\text{Ran}(T)} = \text{Ran}\left(T\right)\,.$$

The proof of Reisz's theorem (see Theorem 6), requires similar minor modifications to accommodate for the fact that we do not have projections and can't compute orthogonal complements of subspaces. However, the slightly weaker statement of Lemma 10 is sufficient. To see this in action, let us prove a part of Reisz's theorem.

Lemma 13. *There exists* $0 < p < \infty$ *such that*

$$\{0\} = \mathcal{N}(T^0) \subsetneq \mathcal{N}(T) \subsetneq \mathcal{N}(T^2) \ldots \mathcal{N}(T^{r-1}) \subsetneq \mathcal{N}(T^p) = \mathcal{N}(T^{p+1}) = \mathcal{N}(T^{p+2}) = \ldots\,, \tag{4}$$

Proof. Suppose not. Suppose that

$$\{0\} = \mathcal{N}(T^0) \subsetneq \mathcal{N}(T) \subsetneq \mathcal{N}(T^2) \ldots \mathcal{N}(T^{r-1}) \subsetneq \mathcal{N}(T^p) \subsetneq \mathcal{N}(T^{p+1}) \subsetneq \ldots$$

Then, by Lemma 10, there exists $\phi_k \in \mathcal{N}(T^{k+1})$ and $\|\phi_k\| = 1$, such that

$$\inf_{\phi \in \mathcal{N}(T^k)} \|\phi_k - \phi\| \geq \frac{1}{2}$$

If $n > m$, then

$$\|A\phi_n - A\phi_m\| = \|T\phi_n - T\phi_m - (\phi_n - \phi_m)\|$$

We note that $T\phi_n - T\phi_m + \phi_m \in \mathcal{N}(T^n)$. Thus

$$\|A\phi_n - A\phi_m\| = \|\phi_n - (T\phi_n - T\phi_m + \phi_m)\| \geq \inf_{\phi \in \mathcal{N}(T^n)} \|\phi_n - \phi\| \geq \frac{1}{2}\,.$$

Thus, $A\phi_n$ cannot have a convergent subsequence, which contradicts the compactness of A

Therefore, there exists some p such that

$$\mathcal{N}(T^p) = \mathcal{N}(T^{p+1})\,.$$

■

Given, the proof of Reisz theorem for Banach spaces, we next proceed to show that a Fredholm operator T is injective if and only if it is bijective, or alternately known as the Fredholm alternative. We observe that the proof outlined for Hilbert spaces in Theorem 7 just carries forward.

Exercise: Show that $\dim(\mathcal{N}(T)) = \dim(\mathcal{N}(T^*))$ if $T \in L(X,X) = I + A$ where A is compact.