

# Random multifractals: negative dimensions and the resulting limitations of the thermodynamic formalism

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The story described in this paper has started with the ‘death or survival’ criterion, which the author published in 1972–1974 and had obtained in 1968 while investigating Kolmogorov’s hypothesis that the turbulent dissipation  $\epsilon(d\mathbf{x})$  in a box is log-normally distributed. Using this criterion, the present paper discusses the concrete significance of negative fractal dimensions. They arise in those random multifractal measures, for which the Cramèr function  $f(\alpha)$  (the ‘spectrum of singularities’) satisfies  $f(\alpha) < 0$  for certain values of  $\alpha$ . It is shown that in that case the strict ‘thermodynamical formalism’ solely involves the form of  $f(\alpha)$  in the range where  $f(\alpha) > 0$ , and concerns three aspects of such measures: (a) the fine-grained multifractal properties, which are non-random and the same for (almost) all realizations; (b) the properties obtained by using the ‘partition function’ formalism; and (c) the ‘typical’ coarse-grained multifractal properties. However, the  $f(\alpha)$ s in the range where  $f(\alpha) > 0$  say nothing about the variability of coarse-grained properties between samples. A description of these fluctuations, hence a fuller multifractal description of the measure, is shown to be provided by the values of  $f(\alpha)$  in the range where  $f(\alpha) < 0$ . We prefer to reserve the term ‘thermodynamic’ for the fine-grained and partition-functional properties, and to say that the coarse-grained properties go beyond the thermodynamics, i.e. are not macroscopic but ‘mesoscopic’.

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## Dedication

While the material in this paper is several stages removed from anything done by Kolmogorov, it is appropriate that it should be dedicated to the memory of Andrei Nikolaievitch. A token of his greatness is that his work spanned fields far removed from one another. I vow special admiration for his several brief forays in highly specialized fields where he could not be more than a transient visitor. His classic K41 theory is of course the occasion of the present collection of papers. His second foray into turbulence, close in content to what I propose to write about, occurred in 1962 and was extremely brief. Kolmogorov’s (1962) paper may even seem an expository report on Obukhov (1962), but in fact puts forward a bold ‘third hypothesis’ of exact log-normality. And careful reading by several authors has revealed profound flaws in this hypothesis. When a giant stumbles, it is safe to expect subtle issues to be involved. We must be grateful to Kolmogorov for having pointed out a path he did not choose to follow himself very far, and for spurring much hard and rewarding work.

# 1. Multifractal measures, Cramèr renormalization, the geometric and analytic forms of scaling, and the pitfalls of the partition function

(a) *The distribution of coarse-grained multifractal measures, and the issue of how the probability densities should be renormalized to ensure asymptotic collapse*

The purpose of this paper is to tackle an issue much more general than log-normality. Nevertheless, it is best to begin with the hypothesis of log-normality, stated in the vocabulary that is today current among physicists, and to use it to describe some subtle ‘anomalies’. Assuming translational invariance, we denote by  $\epsilon(d\mathbf{x})$  the dissipation in a small spatial domain  $d\mathbf{x}$  around a point  $\mathbf{x}$ . The K41 theory postulates that  $\epsilon(d\mathbf{x}) = \bar{\epsilon} dx_1 dx_2 dx_3$ . The K62 theory starts with the postulate that  $\epsilon(d\mathbf{x})$  is random. Being defined for domains  $d\mathbf{x}$ ,  $\epsilon(d\mathbf{x})$  is a random measure.

It is customary in the study of multifractals to write  $\epsilon(d\mathbf{x}) = \bar{\epsilon}|d\mathbf{x}|^\alpha$ . This notation is used in my book (Mandelbrot 1982, p. 372) and in hundreds of more recent references;  $\alpha(d\mathbf{x})$  is a coarse-grained version of the Hölder exponent.

To specify a random function of time, one needs the probability distributions of its increments over different time increments  $dt$ . Similarly, the first things to know about a measure  $\epsilon(d\mathbf{x})$  are the probability distributions for all  $|d\mathbf{x}|$  of the value of  $\epsilon$  (or of  $\alpha$ ) when the domain  $d\mathbf{x}$  is chosen at random. Because of translational invariance, the probability density of  $\alpha$  can be written as  $p(\alpha, d\mathbf{x})$ . Statistical mechanics and the dimensional analysis that underlies all fractal considerations suggest the following question. ‘Can the  $p(\alpha, d\mathbf{x})$ , corresponding to different values of  $|d\mathbf{x}|$  be renormalized so that they collapse into a single expression, at least for  $|d\mathbf{x}| \rightarrow 0$ ?’

The key fact about multifractal measures is that such collapse occurs. The collapsed expression, a function of  $\alpha$ , is usually denoted by  $f(\alpha)$ . For reasons that will become clear in §1d, we shall call it *Cramèr function*. It describes the multifractal properties of the measure  $\epsilon(d\mathbf{x})$ .

The specific goal of this paper is to discuss cases where  $f(\alpha) < 0$  for some values of  $\alpha$ . This will complete a task I undertook many years ago.

(b) *On self similarity, or geometric scaling, versus analytic scaling*

Before we define  $f(\alpha)$ , it is useful to excerpt from the *Manifesto in praise of explicit and visualized geometry* in Mandelbrot (1984). ‘A theme that runs increasingly strongly in my work is that explicit and visualized geometry is important in science and in mathematics. Blind analytic manipulation *is never enough*. Formalisms, however effective in the short run, *are never enough*. Many quantities that have originated in geometry eventually come to be used only in analytic relations; but to forget geometry, and simply identify the fractal dimension  $D$  with an analytic quantity, *is not enough*.’

One may add that the multifractal formalisms, when used mechanically, neglecting their geometric meaning, are not enough. The proof has been provided by the very fact that two schools of thought define the notion of *multifractal* in profoundly different ways.

We must therefore tackle the very distasteful topic of definitions. Informally, ‘fractal sets and multifractal measures are geometric objects such that each small part is very much like a reduced size image of the whole’. Unfortunately, no single formal definition fits, and I use the following realistic semi-formal description. ‘A fractal is a random or non-random set that is geometrically scaling. A multifractal

is a random or non-random measure that is geometrically scaling. Either may be linearly self similar or linearly self affine, or may perhaps fail to fulfil either condition exactly, but come close enough to be handled with the help of the same mathematical tools.'

For other authors, however, 'fractals and multifractals are geometric objects that satisfy diverse analytic scaling relations'.

In the case of fractals on which mass is distributed in fractally homogenous fashion, the mass  $M(R)$  within a radius  $R$  satisfies  $M(R) \sim R^D$ . There is competition between a geometric description of fractal sets and an analytic description based on the above  $M(R)$ . But in practical terms, this has not mattered much.

In the case of multifractals, the situation has proven to be surprisingly different. Since analytic scaling is not directly linked to specific implementations, one may have expected it to be the more general notion. But in fact it has proven to be narrower, and multifractal measures that are geometrically but not analytically scaling prove to be needed to study two topics of fundamental character: turbulence and fractal aggregates. This creates the need for a sharp distinction.

The broad multifractals will be defined as being geometrically scaling. The simplest and best known are the multiplicatively generated multifractals, either non-random (as studied by the school of A. S. Besicovitch at the University of Cambridge) or random (as studied in my works listed in the References). These measures are exactly renormalizable, by design. Diverse analytic scaling relations hold in most cases, but other analytic scaling relations either fail to hold or are of restricted validity.

The narrow multifractals will be defined as being analytically scaling in a strong sense. Therefore, they exclude the broad multifractals for which analytic scaling either fails or is weakened.

At one time, this distinction seemed to lack practical bite, but it is proving increasingly important. In particular, the present paper is devoted to two results. (a) The requirement of analytic scaling excludes the random multiplicative multifractals with  $f(\alpha) < 0$ . (b) Deep differences exist in the extent to which different multifractals are represented by the 'thermodynamical formalism'.

### (c) *The partition function and its pitfalls*

The most widely used analytic scaling relation concerning multifractals involves the partition function  $\chi(q, |d\mathbf{x}|) = \sum \epsilon^q(|d\mathbf{x}|)$ . It concerns a measure  $\epsilon$  that was coarse-grained by boxes of equal side  $|d\mathbf{x}|$ . Divided by a total number of boxes,  $\chi$  is a sample  $q$ th moment of the coarse-grained measure  $\epsilon(d\mathbf{x})$ .

Random or non-random measures constructed by a multiplicative cascade (see §2 below) resulting are geometrically scaling, and the expectation of  $\chi(q, |d\mathbf{x}|)$ , if finite, takes the analytically scaling form  $|d\mathbf{x}|^{\tau(q)}$  (Mandelbrot 1974*a, b*). In the non-random case, 'expectation' is of course to be replaced by actual value. When  $\mu > 0$  for all boxes and the  $|d\mathbf{x}|$  are equal (as we shall suppose throughout), this  $\chi$  is finite. The revival of interest in multifractals first limited to the non-random case. The function  $\tau(q)$  was reintroduced in Hentschel & Procaccia (1983), and Halsey *et al.* (1986) define a (non-random) multifractal as a measure satisfying  $\chi(q, d\mathbf{x}) = |d\mathbf{x}|^{\tau(q)}$ .

When physicists moved from non-random to random multifractals, it seemed reasonable to use the same formalism, reasonable not to envision the possibility that expectations can diverge, and reasonable to expect the sample  $\chi$  to follow the same analytical scaling rule as its expectation. For these reasons, the analytic scaling of the sample  $\chi$  is widely used to *define* the notion of multifractal.

Unfortunately, those apparently reasonable pseudo-ergodic presuppositions were ill-inspired. The sample and population expressions turn out to be very distinct notions in the context of random multifractals. The function  $\tau(q)$  is finite in the non-random case, and is necessarily an increasing function of  $q$ , for all  $qs$ . So is (let me add) the ratio  $\tau(q)/q$ . In the random case, to the contrary, the function  $\tau(q)$  based upon expectations may only be defined in some restricted range,  $q_{\min}^* < q < q_{\max}^*$ , and  $\tau(q)$  and  $\tau(q)/q$  need not be increasing. This difference is a clue to why probabilistically interesting results in Mandelbrot (1974*a, b*) and Kahane & Peyriere (1986) show that many 'physically reasonable' anticipations fail in the case of multifractals. There are many random cases of central importance such that analytical scaling holds for the expectation  $\langle \epsilon^q \rangle$ , where the expectation is carried over the whole ensemble or population of  $\epsilon s$ , but either fails for the sample partition function  $\chi$ , or holds in a weakened form.

(d) *Cramèr renormalization in the log-normal case, and the Cramèr limit function  $C(\alpha)$*

Kolmogorov's hypothesis of log-normality makes no explicit reference to the mechanism that generates the measure  $\epsilon(d\mathbf{x})$ . It merely postulates that there is a parameter  $\mu > 0$ , such that the probability density  $p(\alpha, d\mathbf{x})$  is the gaussian

$$(k/e\pi\mu)^{\frac{1}{2}} \exp\{-k[\alpha - (1 + \frac{1}{2}\mu)]^2/2\mu\}.$$

The factor  $k$  in this expression stands for  $\log_b|d\mathbf{x}|$ , where  $b$  is an integer called basis, to be explained shortly.

It can be seen (Mandelbrot 1974*a, b*, 1989) that this hypothesis is untenable, meaning that a measure having this  $p(\alpha, d\mathbf{x})$  cannot be implemented. The nearest thing conceptually is a multiplicative measure that is constructed using log-normal multipliers, but does not itself involve log-normal  $\epsilon(d\mathbf{x})$ . However, we shall find it useful to list some strange and anomalous consequences of log-normality. Then we proceed to a careful analysis that shows that one consequence is indeed irremediably unacceptable, but another leads to a variety of interesting facts.

As mentioned in §1*a*, the basic issue concerning multifractals is that of data collapse. The above expression for  $p(\alpha, d\mathbf{x})$  can indeed be renormalized in such a way that it ceases to depend as  $|d\mathbf{x}|$ . This is shown directly but only in passing in Mandelbrot (1974*b*, p. 357) and in detail but only implicitly in Frisch & Parisi (1985) and Halsey *et al.* (1986). The key is to perform, what I propose to call, *Cramèr renormalization* by considering the expression

$$C(\alpha, |d\mathbf{x}|) = \ln p(\alpha, d\mathbf{x})/\ln |d\mathbf{x}|.$$

Under the hypothesis of log-normality,

$$\lim_{|d\mathbf{x}| \rightarrow 0} C(\alpha, |d\mathbf{x}|) = -[-(1 + \frac{1}{2}\mu)]^2/2\mu = C(\alpha),$$

by definition of  $C(\alpha)$ . Therefore, Cramèr renormalization yields asymptotic collapse for the log-normal probability distributions. This collapse might have been a serendipitous consequence of log-normality. In the more general cases I have studied and shall summarize in §2,  $\epsilon$  is built by a recursive multiplicative cascade, and Cramèr collapse results from a theorem by Harold Cramèr. In the approach in Halsey *et al.* (1986), a result that is a special case of Cramèr collapse comes out of a mechanical manipulation of symbols. In any event, it remains in each case to deduce  $C(\alpha)$  from the rules governing  $\epsilon$ , and to draw consequences.

## 2. Multiplicative multifractals and the ‘anomalies’ of the log-normal

### $C(\alpha)$ or $f(\alpha)$

#### (a) Background: the binomial multifractals

The binomial measures depend upon a single real parameter  $m_0$ , variously called a multiplier or a mass. One assumes that  $0 < m_0 < 1$  and  $m_0 \neq \frac{1}{2}$ , and one defines  $m_1 = 1 - m_0$ . The basic ‘generating step’ spreads mass over the halves of every dyadic interval, with the relative proportions  $m_0$  and  $m_1$  placed to the left and to the right. Thus, the first stage yields the mass  $m_0$  in  $[0, \frac{1}{2}]$  and the mass  $m_1$  in  $[\frac{1}{2}, 1]$ . After  $k$  stages, let  $\varphi_0$  and  $\varphi_1$  denote the relative frequencies of 0s and 1s in the binary development of  $t = 0.\beta_1\beta_2\dots\beta_k$  written in the counting base  $b = 2$ . With the binomial measure, the dyadic interval  $[dt] = [t, t + 2^{-k}]$  receives the mass  $\mu(dt) = m_0^{k\varphi_0}m_1^{k\varphi_1}$ . It follows that  $\alpha = \alpha(\varphi_0, \varphi_1) = -\varphi_0 \log_2 m_0 - \varphi_1 \log_2 m_1$ .

As to the number of intervals leading to  $\varphi_0$  and  $\varphi_1$ , it is  $N(k, \varphi_0, \varphi_1) = k! / (k\varphi_0! (k\varphi_1)!)$ . Hence, the similarity dimension of the set where this  $\alpha$  is observed is

$$\delta(k, \varphi_0, \varphi_1) = -\ln N(k, \varphi_0, \varphi_1) / \ln(dt) = -\ln [k! / (k\varphi_0! (k\varphi_1)!)] / \ln(dt).$$

For large  $k$ ,  $\lim_{k \rightarrow \infty} \delta(k, \varphi_0, \varphi_1) = -\varphi_0 \log_2 \varphi_0 - \varphi_1 \log_2 \varphi_1$ . Eliminating  $\varphi_0$  and  $\varphi_1$  between  $\alpha$  and  $\delta$ , we obtain a function  $f(\alpha)$  written in parametric form.

It is extremely important to restate  $f(\alpha)$  in terms of rescaled doubly logarithmic plots of probability densities. The binomial measures are, of course, not random, but a random variable appears when a dyadic box of length  $2^{-k}$  is picked at random among the  $2^k$  boxes. Then the Hölder exponent become a random variable, to be denoted by  $H$ . The probability of hitting a prescribed  $\alpha$ , to be denoted by  $Pr\{H = \alpha\}$ , equals  $2^{-k}N(k, \varphi_0, \varphi_1)$  and satisfies

$$C(k, \varphi_0, \varphi_1) = -\ln Pr\{H = \alpha\} / \ln dt = -\ln 2^{-k}N(k, \varphi_0, \varphi_1) / \ln dt.$$

Hence, the Cramèr’s function is  $C(\alpha) = \lim_{k \rightarrow \infty} C(k, \varphi_0, \varphi_1) = f(\alpha) - 1$ . More generally, if  $\epsilon$  is built on a support other than the interval  $[0, 1]$ , one has

$$f(\alpha) = C(\alpha) + \text{dimension of the measure’s support.}$$

This relation is of broad validity, and allows us to evaluate  $f(\alpha)$  via the Cramèr function  $C(\alpha)$ .

#### (b) Random multiplicative measures

The salient feature of the Besicovitch measures in §2a is that each stage of construction multiplies a mass by a non-random factor  $m_0$  or  $m_1$ . To generalize (Mandelbrot 1974a, b), we allow this multiplying factor to be random. For example, consider the base  $b$  box of length  $b^{-1}$  starting at  $t = 0.\beta_000\dots$ . To each  $\beta$  with  $0 \leq \beta \leq b - 1$  corresponds a random multiplier  $M(\beta)$ .

By a repetition of this scheme, the  $b$ -adic box of length  $b^{-k}$  starting at  $t = 0.\beta_1\beta_2\dots\beta_k$  is assigned the mass

$$\mu^k(dt) = M(\beta_1)M(\beta_1, \beta_2)\dots M(\beta_1, \dots, \beta_k) = \Pi M.$$

Here, the successive random multipliers  $M$  for given  $t$  are independent and we shall assume them to be identically distributed.

Plotting  $\mu(dt)$  in logarithmic scale and normalizing by dividing through  $\log_b dt$ , and writing  $-\log_b M = V$ , our box of base  $b$  yields

$$H = \ln \mu(dt) / \ln(dt) = (1/k)[-\log_b M(\beta_1) - \log_b M(\beta_1, \beta_2)\dots] = (1/k) \Sigma V.$$

Thus, the random variable  $H$  is simply the average of  $k$  independent random variables.

### *Strict and average conservation of mass*

To fit the image of 'mass', it is natural to assume some form of conservation. In a conservative construction, the point of coordinates  $M(\beta)$  is a random point in the portion of  $b$ -dimensional space defined by  $M(\beta) \geq 0$  and  $\sum M(\beta) = 1$ . This implies  $M \leq 1$  and  $\langle M \rangle = b^{-1}$ . A variant construction, called canonical (Mandelbrot 1974*a, b*), assumes that the  $M(\beta)$  are statistically independent, without the constraint  $M < 1$ , and that mass is only conserved on the average, meaning that  $\langle M \rangle = b^{-1}$ . These distinctions are important, but it has been shown (Durrett & Liggett 1983) that the rules of dependence between the multipliers do not change the death or survival criterion. The fact that they do not affect the multifractal formalism may even be viewed as demonstrating that this formalism treats as being different measures that may otherwise be important.

### *(c) Standard limit theorems and the Cramèr theorem*

We denote by  $p(v)$  probability density of  $V$  and by  $p_k(v)$  the probability density of  $\sum V$ . In general, the probability density  $kp_k(k\alpha)$  of  $H = (1/k)\sum V$  is impossibly complicated and its limit distribution for  $k \rightarrow \infty$  is necessarily investigated using limit theorems. The plural is needed, because convergence of random variables can be defined in a variety of very different ways, hence there is a multiplicity of limit theorems, all different and each true on its own terms.

The law of large numbers tells us that if  $\langle H \rangle < \infty$  then  $H$  converges to  $\langle H \rangle$ , which implies that  $C(\alpha)$  has its maximum for  $\alpha_0 = \langle H \rangle$ . And the central limit theorem tells us that if  $\langle H^2 \rangle < \infty$  then  $C(\alpha)$  is parabolic in the immediate neighbourhood of  $\alpha_0 = \langle H \rangle$ . In the multifractal context, these results are without interest, the reason being that they give little weight to  $\alpha$ s far from  $\alpha_0$ . To determine  $C(\alpha)$ , one needs the very different 'large deviations theorem' of Harold Cramèr (see Chernoff 1952; Daniels 1954). The Cramèr theory has been rather obscure until now, but it is likely to soon become widely known and used. As  $k \rightarrow \infty$ , the 'local Cramèr theorem' asserts that  $(1/k) \log_b$  (probability density of  $\alpha$ ) converges to a limit  $C(\alpha)$ .

### *The Legendre formalism*

The Cramèr theory also shows that  $f(\alpha) = C(\alpha) - 1$  and  $\tau(q) = -1 - \log_b \langle M^q \rangle$  are linked by the Legendre and inverse Legendre transforms. It is natural that the best proof of Cramèr's result, in Daniels (1954), should use the steepest descent argument.

### *(d) Four anomalies of the log-normal $f(\alpha)$ , and the key to their solution*

The idea has spread that the graph of  $f(\alpha)$  always has the  $\cap$  shape characteristic of the binomial, including the property that  $f(\alpha) > 0$ . In fact, the  $\cap$  shape is only encountered for multifractals that belong to a limited class I have called restricted or narrow. By this standard, the parabolic graph in §1*d* presents numerous 'anomalies'.

### *The $\alpha_1 < 0$ anomaly*

This is in appearance the most absurd of all four. Its background is that the function  $f(\alpha)$  satisfies  $f(\alpha) \leq \alpha$ , and that the equation  $f(\alpha) = \alpha$  has a single root  $\alpha_1$ ,

which for narrow multifractals is both the Hölder exponent and the dimension of the measure theoretical support of the measure  $\mu$ . In the log-normal case, however, one finds that  $\alpha_1 = 1 - \frac{1}{2}\mu$ . It follows that  $\alpha_1 < 0$  when  $\mu > 2$ ; how can a Hölder exponent be negative? This feature is to be discussed in §3*a*.

*The  $\alpha < 0$  anomaly*

This is a second most absurd anomaly, and is encountered for all values of  $\mu$ , even in cases when the first anomaly is absent. Log-normality predicts that (even if  $\mu < 2$  so that  $\alpha_1 > 0$ )  $\alpha$  can take negative values. This feature implies that  $\alpha$  need not be a Hölder exponent. It raises very interesting points that are beyond the scope of this paper. Mandelbrot (1990*c*) shows that the function  $C(\alpha)$  obtained in §1*d* only holds down to  $\alpha = \alpha_{\text{crit}} = -1 + \frac{1}{2}\mu$ ; for  $\alpha < \alpha_{\text{crit}}$ , one must replace this expression by  $C(\alpha) = 2\alpha/\mu - 1$ .

*The  $f(\alpha) < 0$  anomaly*

It is next in apparent absurdity. The log-normal allows  $f(\alpha)$  to be negative for  $\alpha > \alpha_{\text{max}}^* = 1 + \frac{1}{2} + \sqrt{8\mu}$  and  $\alpha < \alpha_{\text{min}}^* = 1 + \frac{1}{2}\mu - \sqrt{8\mu}$ . This fact is observed (without explanation or comment) in Frisch & Parisi (1985). Negative  $f(\alpha)$ s also occur in non-log-normal cases for which the  $\alpha_1 < 0$  and  $\alpha < 0$  anomalies are both absent. This feature is to be discussed in §3*b*.

*Key to the solution of the anomalies*

I propose in this paper to explain the  $\alpha_1 < 0$  and  $f(\alpha) < 0$  anomalies, hence to eliminate them as potential paradoxes. The argument consists in extending the solution of the  $\alpha_1 < 0$  anomaly as given in Mandelbrot (1974*a, b*). The key is a surprising ‘death-or-survival (DOS) criterion’, namely in a mathematical theorem that involves the following feature. When  $X = \lim_{n \rightarrow \infty} X_n$  (with  $X_n \geq 0$ ), it may very well happen that  $\langle X^q \rangle < \lim_{n \rightarrow \infty} \langle X_n^q \rangle$ . A physicist would have expected equality to hold in all cases, and would have considered the known cases of inequality as being ‘pathological counter examples’. The standard counter example given in mathematics textbooks is indeed pathological. It occurs when  $X_n = 0$  with probability  $1 - 1/n$ , and  $X_n = n$  with probability  $1/n$ . Clearly,  $X \equiv 0$  and  $\langle X^q \rangle = 0$  for all  $q > 0$ , but  $\langle X_n^q \rangle = n^{q-1}$ ; therefore  $\langle X^q \rangle < \lim \langle X_n^q \rangle$  for  $q > 1$ . It so happens, however, that the study of random multifractals is full of instances where the moment of a limit need not be the limit of the moments. In particular, a sample average or sum (such as the partition function) may very well behave differently from the corresponding population expectation. When such anomalies are present, the average and the expectation may obey entirely different scaling rules.

**3. The death or survival criterion and thermodynamics**

*(a) The solution of the anomaly of  $\alpha_1 < 0$  involves the death or survival criterion*

To restate this first anomaly. When  $\mu > 2$ , the quantity  $\alpha_1 = 1 - \frac{1}{2}\mu$  satisfies  $\alpha_1 < 0$ . How can it be?

The answer involves the fact that, irrespective of the value of  $\mu$ , there is no such thing as a measure with a log-normal  $\bar{\epsilon}(d\mathbf{x})$ . The nearest thing is a measure obtained by a multiplicative cascade whose multipliers are log-normal random variables. When  $\mu < 2$ , this cascade does indeed generate a non-degenerate measure. This measure is said to ‘survive’.

When  $\mu > 2$ , to the contrary, the limit measure generated by log-normal multiplier is identically zero for all  $|\mathbf{d}\mathbf{x}|$ . It is said to 'die' and  $\alpha_1$  is not a meaningful notion. The calculation that yields  $\alpha_1 < 0$  is just a formal exercise that cannot be made rigorous. Since it does not evaluate any meaningful quantity, the fact that its outcome is negative means nothing.

In general random multiplicative multifractals, the DOS criterion involves the quantity  $\alpha_1 = \langle M \log_b M \rangle$ . When  $\alpha_1 > 0$ , the measure survives; when  $\alpha_1 < 0$ , the measure dies. When the cascade preserves mass, it necessarily survives, because  $\alpha_1 > 0$  and  $\sum \epsilon(\mathbf{d}\mathbf{x})$ . In all other cascades with  $\alpha_1 > 0$ , the sum  $\sum \epsilon(\mathbf{d}\mathbf{x})$  converges to a limit random variable (Mandelbrot 1974*a, b*; Kahane & Peyriere 1976; Durrett & Liggett 1983). When the cascade starts with measure 1, the limit mass is currently denoted (Mandelbrot 1990*c*) by the letter  $\Omega$ .

(b) To solve the  $f(\alpha) < 0$  anomaly, it suffices to apply the death or survival criterion to the multifractal  $\epsilon^{(q)}$ , which is defined as being constructed with the multipliers  $M^{(q)} = M^q/b\langle M^q \rangle = M^{qb^{-\tau(q)}}$

Indeed, taking  $|\mathbf{d}\mathbf{x}| = b^{-k}$ , the expression  $\chi(q, \mathbf{d}\mathbf{x}) = \sum \epsilon_i^q(\mathbf{d}\mathbf{x})$  is simply the product by  $(b^k)^{\tau(q)}$  of the sample sum of the multifractal  $\epsilon^{(q)}$  as defined in the title above is a random variable and we know from §3*a* that its behaviour depends on the sign of the quantity  $\alpha_1^{(q)}$ , defined as the  $\alpha_1$  relative to the measure  $\epsilon^{(q)}$ .

Before we pursue, let us point out that  $\alpha_1^{(q)} > 0$  when  $q$  is in the range  $q_{\min}^* < q < q_{\max}^*$  which corresponds  $f(\alpha) > 0$ , hence to  $\alpha_{\min}^* < \alpha < \alpha_{\max}^*$ ; when  $f(\alpha) < 0$ , one has  $\alpha_1^{(q)} < 0$ . Indeed, define  $f^{(q)}(\alpha)$  as the function  $f(\alpha)$  relative to  $\epsilon^{(q)}$ . To obtain geometrically the graph of  $f^{(q)}(\alpha)$ , one takes the following steps. (a) Expand  $\alpha$  and  $f(\alpha)$  horizontally in the ratio  $q$ , obtaining a new variable  $\tilde{\alpha}^{(q)}$  and a new function  $\tilde{f}^{(q)}(\alpha)$ . (b) Trace the tangent of slope 1 of  $\tilde{f}^{(q)}(\alpha)$ . This is the expanded form of the tangent of slope  $q$  of  $f(\alpha)$ . (c) Reset the origin of  $\tilde{\alpha}^{(q)}$  at the point where the tangent intersects the axis  $f = 0$ . This will define the quantity  $\alpha^{(q)}$ . We can now apply the death or survival criterion to  $M^{(q)}$ .

When  $\alpha_1^{(q)} < 0$ , the DOS criterion tells us the measure constructed using the multipliers  $M^{(q)}$ , i.e. the sequence of the quantities  $\epsilon^{(q)}$ , converges to a non-degenerate random limit, which is the variable  $\Omega^{(q)}$  corresponding to the new multiplier  $M^{(q)}$ . Therefore, in agreement with general expectation, the sample sum  $\chi(q, \mathbf{d}\mathbf{x})$  can serve to estimate the population expectation  $b^k \langle \epsilon^q(\mathbf{d}\mathbf{x}) \rangle$ . In particular, the sample sum and the population expectation follow precisely the same analytic scaling rule, namely  $\chi(q, \mathbf{d}\mathbf{x}) = b^{k\tau(q)} \Omega^{(q)}$ . Therefore, the sample  $\tau(q)$  is identical to the population  $\tau(q)$ .

When  $\alpha_1^{(q)} < 0$ , to the contrary, the sample sum of  $\epsilon^{(q)}$  converges to zero. The unexpected consequence is that geometric scaling of  $\mu$  fails to imply that the sample sum  $\chi(q, \mathbf{d}\mathbf{x})$  is analytically scaling. However, an elementary but lengthy argument, for which we have no space, shows for  $q < q_{\min}^*$  the sample sum is asymptotically scaling for  $\mathbf{d}\mathbf{x} \rightarrow 0$ , with the scaling exponent  $\tilde{\tau}(q) = q\alpha_{\min}^*$ , while for  $q > q_{\max}^*$ , the asymptotic scaling exponent is  $\tilde{\tau}(q) = q\alpha_{\max}^*$ . The Legendre transform of this  $\tilde{\tau}(q)$  happens to be simply the portion of  $f(\alpha)$  where  $f(\alpha) > 0$ . The portion where  $f(\alpha) < 0$  plays no role whatsoever in the asymptotic study of  $\chi$ . However, this  $\tilde{\tau}(q)$  is only defined asymptotically for  $\mathbf{d}\mathbf{x} \rightarrow 0$ . This may explain the reports that  $f(\alpha) < 0$  has been obtained from the partition function, and could not be accounted for.

We have carried out extensive computations to test this prediction; the results are very striking. The reader is advised to test them again, for example using the measures described in Mandelbrot (1989*b*, 1990*a*).



(c) *The notion of thermodynamic formalism and its limitations*

In every approach to multifractals, the actual manipulations that link  $f(\alpha)$  to  $\tau(q)$  are directly borrowed from thermodynamics. For example, the probabilists who work with large deviations view the Cramèr theory as being thermodynamical. Also, when asked to introduce  $f(\alpha)$  and  $\tau(q)$  in the simplest context, I prefer to use the Lagrange multipliers path towards the thermodynamics (Mandelbrot 1989*a*, §6). As to the approach in Halsey *et al.* (1986), it uses steepest descents, hence (without saying so) uses the Darwin–Fowler approach of thermodynamics.

However, noticing analogies between formalisms is not enough. As we dig deeper, we find ambiguities in the scope that should be given to the notion of thermodynamic property.

One might have argued that it should cover everything that deals with asymptotics. In the present case, the relevant asymptotics is the Cramèr theory. With this definition, one would describe the whole function  $f(\alpha)$  as thermodynamic, including the part in which  $f(\alpha) < 0$ . In my opinion, however, this view would be inclusive to excess. Even though the values of  $f(\alpha) < 0$  are obtained by a limit theorem, the actual estimation of  $f(\alpha) < 0$  must rely upon methods that involve preasymptotics; for example it can use the method based on actual histograms within ‘supersamples’ (Mandelbrot 1989*a*, §4.3–4.5; 1990*c*, §5.8).

I would more readily argue that the notion thermodynamic property should be reserved to considerations that involve the partition function. This restricts  $f(\alpha)$  to the values  $f(\alpha) > 0$ .

Furthermore, my earlier studies of  $f(\alpha) < 0$  (Mandelbrot 1989*a*, §4.3–4.5; 1990*c*, §5.8) show that the portion  $f(\alpha) > 0$  can serve to define the notion of ‘typical’ multifractal properties in the case when they are genuinely random, which happens to occur when  $d\mathbf{x}$  is small but positive.

The same studies can also be pushed (but we have no room here for this task) to show that, as  $|d\mathbf{x}| \rightarrow 0$ , the multifractal properties of  $\epsilon(d\mathbf{x})$  cease to be random, and that they are fully determined by the portion  $f(\alpha) > 0$ .

It remains to give a name to the properties that depend on the portion where  $f(\alpha)$ , hence are not strictly thermodynamical. I propose to call them mesoscopic.

There is no room here to summarize the properties of multifractals with a left-sided  $f(\alpha)$  (see, for example, Mandelbrot 1990*b*). They also involve mesoscopic effects, which are of very different kind and of equally great practical importance.

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