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NEGATIVE FRACTAL DIMENSIONS AND MULTIFRACTALS

Benoit B. MANDELBROT
Physics Department, IBM T.J. Watson Research Center, Yorktown Heights, NY 10598, USA and Mathematics Department, Yale University, New Haven, CT 06520, USA

A new notion of fractal dimension is defined. When it is positive, it effectively falls back on known definitions. But its motivating virtue is that it can take negative values, which measure usefully the degree of emptiness of empty sets. The main use concerns random multifractals for which $f(\alpha) < 0$ for some α's. The positive $f(\alpha)$ are shown to define a "typical" distribution of the measure, while the negative $f(\alpha)$ rule the sampling variability. Negative dimensions are best investigated using "supersamples." Applications are to turbulence and to DLA.

1. Introduction

It is a pleasure to attend this fifth consecutive Stat Phys meeting. The topic of multifractals was already prominent in our previous Invited Lectures [1, 2]. And negative fractal dimensions is briefly announced in [2], as one of two separate aspects of dimension, that are *latent* (= hidden, but present). Lately, many authors have added much to the topic of multifractals, and it has greatly changed (though our early papers may not yet be exhausted). Despite these advances, however, even the most basic aspects of multifractality continue to present features that deserve further research.

The link between the two topics in the title came to focus recently, and it is elementary, that is, should be widely used. First, we develop negative dimension as a new notion, and introduce those physicists who have already become used to life in fractional dimension to the charms of negative dimension, and to its inevitability. Secondly, we study some random multifractal measures, for which $f(\alpha) < 0$ for some α's. In broad outline, positive $f(\alpha)$'s define and describe a *typical* distribution of a random fractal measure, and negative $f(\alpha)$'s describe the *fluctuations* one may expect in a finite size sample.

A look back. Our first survey of multifractals for the physicist, in [1], came out before its time, but after a 1975 survey for the general reader [3], and of course our original papers (e.g., [4]) had tackled turbulence using multifractals (without the word); see also [5, 6]. We would like to quote from the 1977 text [1] using current notation: [The physics of] "critical points [involves many] exponents .... [The same holds for] intermittent turbulence .... Scaling is

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compatible with inequality between exponents, while equality demands an especially strong form of scaling ... [and occurs] when the graph of $\tau(q)$ reduces to a straight line .... Geometrically speaking, a special and unique virtue of a straight $\tau(q)$ [is that] dissipation [is] homogeneous over a closed subset of space to which it is restricted .... Every other $\tau(q)$ implies that the bulk of dissipation is homogeneous over a small set, but [there is] one remainder spread around everywhere else and another remainder concentrated in sharp peaks .... Different moments of the dissipation turn out to be very much affected by one or by the other remainder."

2. A generalized "latent" fractal dimension can be negative. Examples

First, we restate the familiar "intersection rule" for dimensions, and its equally familiar exception. Then we restate a suggestion made in [2], and buttress it by new arguments and concrete illustrations.

Generic intersection rule [5]. Take two sets S_1 and S_2 (either Euclidean or fractal) in a Euclidean space of ordinary (embedding) dimension E. Denote their codimensions by $E - \dim(S_1)$ and $E - \dim(S_2)$. "Generically," the rule is that the intersection S of S_1 and S_2 has the codimension

$$E - \dim(S) = E - \dim(S_1) + E - \dim(S_2) .$$

Major exception to the rule. When $E - \dim(S_1) + E - \dim(S_2) > E$, its value does not matter: the intersection S is generically empty.

A way to redefine dimension, which avoids this exception, and simplifies but enriches the intersection rule. The example of points, lines, planes and the like. Compare the intersection of two lines and the intersection of a line by a plane. Both sets are "generically" of dimension 0, in agreement with the intersection rule and its exception. Yet, one would like to discriminate more finely between these various ways of being of dimension 0, by expressing numerically the idea that the intersection of two lines is "emptier" than the intersection of a line by a plane. If one could get rid of the exception to the intersection rule, one may perhaps be allowed to say that these two sets have the dimensions -1 and 0.

This loose idea of "latency" can indeed be given precise and down-to-earth meanings. The root of the explanation is that one cannot observe an unbounded space, with strict points, lines or planes, only a bounded "window" of space, with small blobs, thin sticks and thin shells.

A generalized box dimension that is -1. A set of (Euclidean or fractal) dimension D_B requires $N(b) \sim b^{D_B}$ boxes of side $r = b^{-1}$ to be covered. The familiar box dimension D_B simply measures *the rate of increase of* $N(b)$ with b.

One should be able to generalize D_B as describing the rate of either increase or decrease of "something that is like $N(b)$." This something could not be a number of boxes, which is an integer. But let us show that it could be $\langle N \rangle$.

To simplify the algebra, focus on a point and a line in the plane. Start with a square window of side L that includes a point-like blob of side $1/b$ and a line-like strip of width $1/b$. When the strip intersects the blob, $N=1$; otherwise, $N=0$. Intersection occurs when the distance between the point and the line is $<b$, which happens with probability $\sim b/L$. Thus, $\langle N \rangle \sim L/b$, and for large L/b we obtain the value

$$D_B = \log(1/b)/\log b = -1 .$$

A generalized sausage codimension that is 3 *in the plane*. The familiar sausage of S is the set of points which lie within a distance ε of a point in S. The sausage of the union of S_1 and S_2, is therefore the union of the sausages of S_1 and S_2. But what about the intersection? When S_1 and S_2 intersect, the intersection of the sausage and the sausage of the intersection scale in the same way as $\varepsilon \to 0$. When the sets S_1 and S_2 fail to intersect, only the intersection of the sausages continues to be defined. In the present example of a point and a line in the plane, its area is $\sim \varepsilon^2$ with a probability $\sim \varepsilon/L$, and 0 otherwise. Hence the expected area of the intersection is $\sim \varepsilon^3/L$. The exponent is a generalized sausage codimension. Its value is $D_S = 3$, which confirms $D = -1$.

Randomness is central to allowing this generalized dimension to become negative. The above expressions for D_B and D_S avoid the fact that N is an integer, by not referring to a single well defined construction, rather to a random ensemble or population of constructions. When D generalized in this way is <0, D says nothing about any specific set, but it describes and classifies a generic reason why a set happens to be empty.

In the past, we had felt that attempts to define a dimension for population were confusing. More important, they fulfilled no need, but we shall see that now a need has been created by the multifractals.

The preceding reasoning extends to fractal sets. The example of the birth-and-death cascades on the interval [0, 1]. The kth cascade stage begins with dyadic cells of length 2^{-k}. Each cell is subdivided into 2 dyadic halves, and each dyadic half either "lives," with the probability $p<1$, or "dies," with the probability $1-p$. One defines a "birth and death process" [5, chapter 23] by thinking of mother cells as dying and giving birth to N daughter cells. Here $N=0$ with the probability $(1-p)^2$, $N=1$ with the probability $2p(1-p)$, and $N=2$ with the probability p^2. Thus, $\langle N \rangle = 2p$.

When $\langle N \rangle > 1$, hence $D = \log_2 \langle N \rangle > 0$, it is known that this process has a positive probability of generating a non-empty set one can call a birth and death fractal dust. D is the value of all useful forms of fractal dimension. But it is also known that the "bloodline" can die out, with probability one when $D<0$, and with probability between 0 and 1 when $D>0$. We now propose to say that $D=\log_2\langle N\rangle$ in all cases, even if $D<0$.

Imbedding a "dead" birth and death fractal set as a one dimensional cut through a proper "living" birth and death set. Start with the E-dimensional cube $[0, 1]^E$, and apply the same birth and death process to each its 2^E sub cubes. Now, $\langle N_E\rangle = 2^E p$. Therefore, however small the value of p, one can choose E to be large enough to insure $2^E p>1$, hence $D_E>0$. Every birth and death set on the line can be interpreted as a cut.

Limitations of the Hausdorff–Besicovitch dimension D_{HB}. The proposed generalizations add to the notion of fractal dimension. Complication increases because one tries to tame yet another facet of reality, using geometric information that is available but otherwise discarded. To extend D_{HB} to negative values would be impossible and randomness is totally foreign to D_{HB}. In the case of random sets, D_{HB} applies to samples and not to ensembles or populations. But we know that, as the intuitive contents of fractal dimension expands, it becomes increasingly clear that physics requires more than D_{HB}. Hausdorff and Besicovitch have provided us with a nearly ready-made notion that could be made into a useful tool of physics. But now this tool has proven to lack versatility and to be hard to use, and it keeps being thoroughly modified and diversified.

3. Scaling requirements that define a self-similar multifractal

The remainder of this paper shows how negative dimensions help understand the self-similar fractal *measures* called multifractals. When a measure $\mu(\mathrm{d}t)$ is carried by $[0, 1]$, it is widely thought that it is self-similar when it satisfies two requirements. A) There exists an exponent α, function of t, such that $\mu(\mathrm{d}t)$ is of the order of $(\mathrm{d}t)^\alpha$. B) The set of values of t where α takes a certain value is a fractal. That is, $N(\mathrm{d}t)$, defined as the number of intervals characterized by α, is of the order of $(\mathrm{d}t)^{-f(\alpha)}$.

Refs. [7, 8] obtain $f(\alpha)$ as the Legendre transform of $\Sigma\, \mu^q(\mathrm{d}t)$. However, their conceptual framework fails to generalize to random multifractals. To handle randomness, we restate requirement B) in terms of the quantity

$$\frac{N(\mathrm{d}t)}{(1/\mathrm{d}t)} = \mathrm{d}t\, N(\mathrm{d}t) = (\mathrm{d}t)^{-f(\alpha)+1} = (\mathrm{d}t)^{-\rho(\alpha)}\,, \quad \text{with} \quad \rho(\alpha) = f(\alpha) - 1\,.$$

This is the relative frequency, among $1/\mathrm{d}t$ intervals of length $\mathrm{d}t$, of those intervals in which a certain value of α is observed. Thinking of this frequency as a probability, the requirement $N(\mathrm{d}t) \sim (\mathrm{d}t)^{-f(\alpha)}$ translates into

$$\frac{\log(\text{probability of } \alpha)}{\log \mathrm{d}t} \text{ is a function } \rho(\alpha) .$$

The last expression can be said to involve a plot of the probability distribution of the random measure $\mu(\mathrm{d}t)$ in renormalized doubly logarithmic coordinates. *First*, $\mu(\mathrm{d}t)$ is replaced by $\log \mu(\mathrm{d}t)$ and renormalized by $\log \mathrm{d}t$, to obtain an abscissa that is a random variable A. The use of the upper case Greek A (capital alpha) follows the custom of the probabilists: A denotes a random variable whose values are denoted by α. *Second*, we work on the probability density $p(\alpha)$ of A; we replace it by $\log p(\alpha)$ and again (to renormalize) we divide it by $\log \mathrm{d}t$.

A definition of generalized self-similar multifractals. We shall say that a measure $\mu(\mathrm{d}t)$ on $[0, 1]$ is a self-similar multifractal if, as $\mathrm{d}t \to 0$, the transformed and renormalized density $\log p(\alpha)/\log \mathrm{d}t$ has a limit $\rho(\alpha)$.

The function $f(\alpha)$ is then defined starting from $\rho(\alpha)$, as $f(\alpha) = \rho(\alpha) + 1$.

4. A multiplicative cascade, and a simple "trio" multifractal

Construction. Let a cascade begin with mass equal to 1, uniformly spread over $[0, 1]$, and let the kth cascade stage share the mass in a cell of length 2^{-k} between two halves of length 2^{-k-1} in either of the following 3 ways: half and half; in the ratios m_0 and $m_1 = 1 - m_0$ with $m_0 \geqslant \frac{1}{2}$, or in the ratios m_1 and m_0. This multiplies the mass in either half by a random variable M that is m_0, $\frac{1}{2}$, or $m_1 = 1 - m_0$, with the respective probabilities

$$\Pr\{M = m_0\} = \tfrac{1}{8} , \quad \Pr\{M = \tfrac{1}{2}\} = \tfrac{6}{8} , \quad \text{and} \quad \Pr\{M = m_1\} = \tfrac{1}{8} .$$

By a repetition of this scheme, the dyadic cell $[\mathrm{d}t]$ of length $\mathrm{d}t = 2^{-k}$ that starts at $t = 0 \cdot \eta_1 \eta_2 \cdots \eta_k$ determines k identically distributed and independent random multiplier variables M, and one has

$$\mu_1(\mathrm{d}t) = M(\eta_1) M(\eta_1, \eta_2) \cdots M(\eta_1, \ldots, \eta_k) ,$$

$$\mathrm{A}_k = \log \mu_1(\mathrm{d}t)/\log \mathrm{d}t = -(1/k)[\log_2 M(\eta_1) + \log_2 M(\eta_1, \eta_2) \cdots] .$$

To know $\mu_1(\mathrm{d}t)$ is to know for all k the probability density $p_k(\alpha)$ of A_k.

The main finding is suggested by an argument in section 3: the expression $\rho_k(\alpha) = (1/k) \log_2 p_k(\alpha)$ converges for large k to

$$\rho(\alpha) = -1 + \text{the Legendre transform of } \tau(q) = -\log_2 \langle M^q \rangle - 1\,.$$

The proof (which cannot be given here) uses a remarkable old (1937) theorem by Harald Cramèr, which has led to what probabilists call the theory of large deviations [9], and must not remain unknown to physicists.

The novelty that $f_1(\alpha) < 0$ is traced to our having defined $\tau(q)$ through the expectation $\langle M^q \rangle$, instead of the non-averaged "partition function" $\Sigma\, \mu_1^q(\mathrm{d}t)$.

The resulting $\rho(\alpha)$ is drawn on fig. 1. First remark: α ranges from $\alpha_{\min} = -\log_2 m_0$ to $\alpha_{\max} = -\log_2 m_1$, which is a familiar feature in binomial measures. Second feature: $f_1(\alpha)$ ranges from -2 to 1, which brings in the striking novelty that $f_1(\alpha)$ is negative for $\alpha^*_{\max} < \alpha \leqslant \alpha_{\max}$ and $\alpha_{\min} \leqslant \alpha < \alpha^*_{\min}$. This can never happen in the Frisch–Parisi context, and destroys their "steepest descents" justification of the Legendre transform.

Imbedding the trio measure as a one dimensional cut through a "conventional" multinomial measure in 3d space. Now consider the following cascade on a cube subdivided into $2^3 = 8$ cubes. Each cascade stage begins with the mass m;

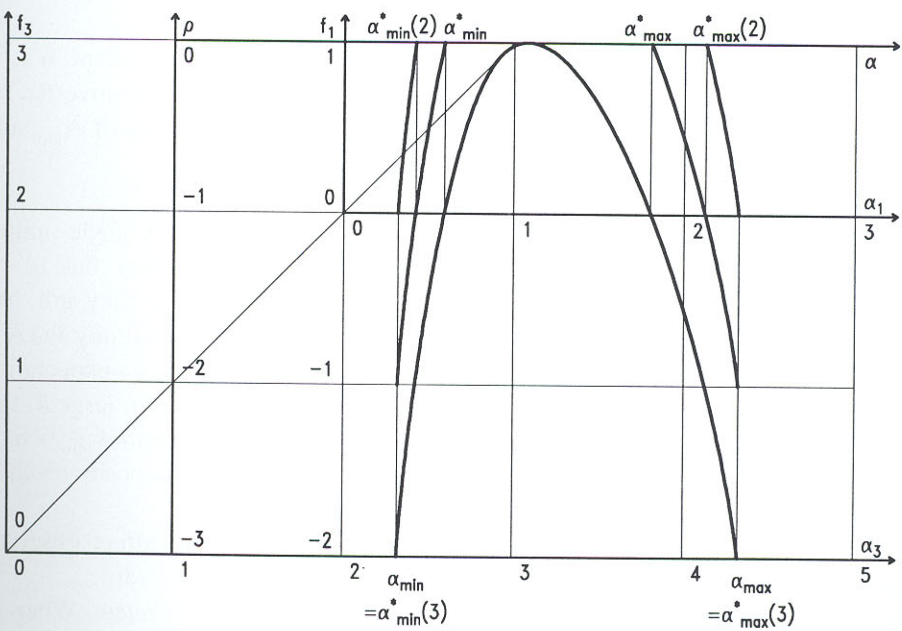

Fig. 1. The $\rho(\alpha)$, $f_1(\alpha)$ and $f_3(\alpha)$ functions of the measures of $\mu_1(\mathrm{d}t)$ and $\mu_3(\mathrm{d}t)$ in the text, with $m_0 = 0.2$. To make it possible for a single "ogive" (bold curve) to represent all three functions, three distinct sets of coordinate axes are drawn (bold lines), for f_3, ρ, and f_1. The additional curves to the right (resp., to the left) of the ogive relate to the approximate distributions of $A_{\max}(2)$ and $A_{\max}(3)$ (resp., of $A_{\min}(2)$ and $A_{\min}(3)$), as defined in the text. The value of $\alpha^*_{\max}$ is a gross underestimate of $\alpha_{\max}$.

the share of a randomly picked "hot" cube is $mm_0/8$, the share of a randomly picked "cold" cube is $mm_1/8$ and the share of each of the 6 remaining cubes is $m/8$. This construction leads to a spatial multifractal $\mu_3(\mathrm{d}\boldsymbol{x})$ largely due to Besicovitch and called multinomial [10, 11]. The Legendre formalism enters (in the familiar Frisch–Parisi interpretation [7, 8]) to yield $f_3(\alpha) = \rho(\alpha) - 3 = f_1(\alpha) + 2$.

That is, $\rho(\alpha) = f_E(\alpha) - E$ is the same for $E = 1$ and $E = 3$, and ranges from -3 to 0. Furthermore, α itself depends on E, but $\alpha - E$ does not.

The conventional interpretation of $f_1(\alpha)$. It is known that $f_3(\alpha)$ is the fractal dimension of the 3d set where the Hölder exponent of $\mu_3(\mathrm{d}\boldsymbol{x})$ is α. Applying the intersection rule with its exception (section 2) to multifractals, the positive values of $f_3(\alpha) - 2$ are the dimensions of the sets where the Hölder α of $\mu_1(\mathrm{d}t)$ is α. The negative $f_3(\alpha) - 2$ simply "saturate" to 0. We claim that this is a waste of valuable information.

5. Typical behavior and variability of $\mu_1(\mathrm{d}t)$: samples and supersamples

Let us indeed show the following. Even when there are latent α's, a "typical" first approximation is defined and determined by the positive $f(\alpha)$'s, and $\mathrm{A}_{\min} \sim \alpha^*_{\min}$ and $\mathrm{A}_{\max} \sim \alpha^*_{\max}$. But the sampling distributions of $\mathrm{A}_{\min}$ and $\mathrm{A}_{\max}$ are mostly ruled by the latent portions of $f(\alpha)$.

Though $b = 2$, the sequel is written in terms of arbitrary b.

The "*typical*" range $[\mathrm{A}_{\min}, \mathrm{A}_{\max}]$ *when some* α*'s are latent*. A single sample with $\mathrm{d}t = b^{-k}$ yields b^k values of $\mu_1(\mathrm{d}t)$. We know that they fail to be statistically independent, but it is useful to first suppose that they are. An heuristic argument then suggests that one can estimate the probability $\Pr\{\mathrm{A} = \alpha\}$ if, and only if, the number of occurrences of this value α has an expectation at least equal to 1. This yields the condition $\Pr\{\mathrm{A} = \alpha\} b^k \geqslant 1$. For large k, this reads $b^{k\rho(\alpha)+k} \geqslant 1$, yielding $\rho(\alpha) \geqslant -1$ or $f_1(\alpha) \geqslant 0$. The range from $\mathrm{A}_{\min} \sim \alpha^*_{\min}$ to $\mathrm{A}_{\max} \sim \alpha^*_{\max}$, is to be called "typical." It is determined by the positive values of $f(\alpha)$. *It grossly underestimates the true range* $[\alpha_{\min}, \alpha_{\max}]$.

If this were the last word, negative $f_1(\alpha)$'s would fail to affect observed $\mu_1(\mathrm{d}t)$, hence could not be estimated. But this is *not* the last word.

The distribution of the range $[\mathrm{A}_{\min}, \mathrm{A}_{\max}]$ *when some* α*'s are latent*. When b^k data are statistically independent, $\mathrm{A}_{\max}$ is given by the theory of "extreme values" of probability theory. One has $\Pr\{\mathrm{A}_{\max} < \alpha\} = [\Pr\{\mathrm{A} < \alpha\}]^{b^k}$, because the inequality $\{\mathrm{A}_{\max} < \alpha\}$ holds if, and only if, b^k independent inequalities of the form $\{\mathrm{A} < \alpha\}$ hold simultaneously. One can show, for $\alpha > \langle \mathrm{A} \rangle$, that $\Pr\{\mathrm{A} > \alpha\}$ is $\sim$ the probability density $b^{k\rho(\alpha)}$. Therefore

$$\Pr\{A_{\max} > \alpha\} \sim 1 - [1 - b^{k\rho(\alpha)}]^{b^k} \sim 1 - \exp\{-b^{k[\rho(\alpha)+1]}\}$$

$$\begin{cases} \sim 1 & \text{when } \rho(\alpha) < -1, \text{ i.e., } \alpha < \alpha^*_{\max} \\ \sim b^{k[\rho(\alpha)+1]} & \text{when } \rho(\alpha) > -1, \text{ i.e., } \alpha > \alpha^*_{\max} \,. \end{cases}$$

First conclusion: for every ε, $\Pr\{|A_{\max} - \alpha^*_{\max}| > \varepsilon\} \to 0$ as $k \to \infty$. This sharpens slightly the notion of "typical range." But adding the fact that $\Pr\{A_{\max} > \alpha\} \sim$ probability density $p(\alpha_{\max})$ yields far more, namely

$$\log p(\alpha_{\max}) / \log \mathrm{d}t \to \rho(\alpha) + 1 \,.$$

That is, the "$\rho(\alpha)$ function" of $A_{\max}$ is simply $\rho_{\max}(\alpha) = \rho(\alpha) + 1$.

Overall graphical expression of the above results. Translate the $\rho(\alpha)$ function of A up by unity; discard the middle portion; denote the portion to the right as $\rho_{\max}(\alpha)$ and assign it to $A_{\max}$; denote the position to the left as $\rho_{\min}(\alpha)$ and assign it to $A_{\min}$ (Observe that the $\tau(q)$ functions corresponding to the limits of $\Pr\{A_{\min} > \alpha\}$ and $\Pr\{A_{\max} < \alpha\}$ are "anomalous.")

Reason to expect the preceding results to be exact. A rigorous distribution of $A_{\max}$ is not available now, but a closely related problem has been fully investigated in the literature [12], and it yields the same result.

The notion of "supersample" made of an increasing number $b^{k(N-1)}$ of independent samples of a random measure μ. Having squeezed information about $f(\alpha)$ from a single sample of non-independent data, we pool data from N statistically independent realizations into a "supersample." We write $N = b^{(E-1)k}$, hence the supersample size is b^{Ek}, because we think of E as an embedding dimension. This odd notation is only justified after we show that the effectiveness of supersampling is measured by $E = 1 + (1/k) \log_b N$.

An estimate of the range of variability of $[A_{\min}(E), A_{\max}(E)]$ *within a supersample.* Define $A_{\max}(E)$ as the largest value of in a sample of size $N = b^{E-1}$. Under the assumption that the b^{kE} data are statistically independent, an already used rough argument suggests that it may be possible to estimte $\Pr\{A_{\max}(E) = \alpha\}$ if, and only if, $\Pr\{A = \alpha\} b^{Ek} \geq 1$, that is, $b^{k\rho(\alpha)+kE} \geq 1$. This yields a condition much weaker than $\rho(\alpha) \geq -1$, namely $\rho(\alpha) \geq -E$ or $-E + 1 \leq f(\alpha) < 0$.

Distribution of the range $[A_{\min}(E), A_{\max}(E)]$ *under the assumption that the* b^{Ek} *values in our sample are statistically independent.* One finds

$$\Pr\{A_{\max}(E) > \alpha\} \sim 1 - [1 - b^{k\rho(\alpha)}]^{Nb^k} \sim 1 - \exp\{-b^{k[\rho(\alpha)+E]}\}$$

$$\begin{cases} \sim 1 & \text{when } \rho(\alpha) > -E, \text{ i.e., } \alpha < \alpha^*_{\max}(E) \\ \sim b^{k[\rho(\alpha)+E]} & \text{when } \rho(\alpha) < -E, \text{ i.e., } \alpha > \alpha^*_{\max}(E) \,. \end{cases}$$

Conclusion: As the supersample size grows, the variability of the range $[A_{min}(E), A_{max}(E)]$ is controlled by increasingly latent portions of $f(\alpha)$.

6. Additional comments

On the distinction between the asymptotic and the preasymptotic roles of the function $f(\alpha)$. The function $f(\alpha)$ plays two roles [11], to be recalled momentarily. They are hard to tell apart when $f(\alpha) \geqslant 0$ for all $\alpha > 0$, but section 5 has shown that they separate sharply when some $f(\alpha) < 0$. The first role, that of a "spectrum of singularities," is an asymptotic property meaningful only for $\mathrm{d}t \to 0$. The second and less widely appreciated role of $f(\alpha)$ concerns its relevance to the histograms of the measure $\mathrm{d}\mu(\mathrm{d}t)$ for various $\mathrm{d}t > 0$.

On goals. One reason to estimate fractal dimensions and $f(\alpha)$'s is to do physics. Another reason is to compare data sets and theories with each other. When the concern is with (say) DLA or turbulence data, one does not estimate $f(\alpha)$ because dimensions are intrinsically interesting but, because $f(\alpha)$ is a possible window of a generating mechanism. It is important, therefore, that the positive $f(\alpha)$ fail to exhaust all that one should attempt to extract from the data. One can tell more about the generating process from the negative $f(\alpha)$, which need not remain latent = hidden. They tell us what to expect from other random samples of the same process. They give us therefore, firmer grounds for the comparison between two distinct sets of data, or for the proper fitting of a model to the data.

The results in section 5 fail to be universal, which is an added complication. It has long been known that when the fractal dimension of a set is known, the set is specified very partially. Section 4 of [13] shows that a seemingly mild modification of $\mu_1(\mathrm{d}t)$ yields a multifractal to which the results in the present section 5 fail to apply.

The Legendre transform. Of course, the two roles of $f(\alpha)$ are indissolubly linked, and the only way to reach $f(\alpha)$ empirically is to estimate it by processing the histograms. One *must not* view the Legendre transforms as providing a definition of $f(\alpha)$. It is only one particular method, among several other, for estimating $f(\alpha)$ from the data. Other methods make more direct use of the histograms [14, 15]. In addition, a brutal use of computer programs embodying the Legendre method when latent α are present yields estimates of $f(\alpha)$ that are sharply sample dependent, and always yields $f > 0$, even when the problem demands negative f's [16, 17].

Many authors have observed that one can obtain $f(\alpha) < 0$ by first averaging $\Sigma\ \mu^q(\mathrm{d}t)$ over supersamples. But such averaging cannot be justified by the Frisch–Parisi argument.

Conclusion and applications. In sum, there is more to a multifractal than $f(\alpha)$, but this only adds to the reasons for studying $f(\alpha)$ as fully as possible. Negative $f(\alpha)$'s give every sign of being essential in the study of turbulence and of DLA, and of all other phenomena that exhibit very high sample variability.

Fig. 1 is very suggestive of the situation that prevails for cuts through a field of turbulent dissipation, and [18, 19] have already demonstrated the practical relevance of the viewpoints described in this paper.

For DLA [20], a more closely illustrated example is provided in [13] (of which the present work is, otherwise, a summary).

Detailed recent treatments of our approach to multifractals, which started with [4], are found in [10, 11, 13, 21].

References

[1] B.B. Mandelbrot, in: Statistical Physics 13, Int. IUPAP Conf. (Haifa, 1977). D. Cabib, C.G. Kuper and I. Riess, eds., Annals of the Israel Physical Society (Adam Hilger, Bristol, 1978), p. 225.

[2] B.B. Mandelbrot, in: Statistical Physics 15, Int. IUPAP Conf. (Edinburgh, 1983), D. Wallace, ed., J. Stat. Phys. 34 (1984) 895.

[3] B.B. Mandelbrot, Les objets fractals: forme, hasard et dimension (Flammarion, Paris, 1975, 1984, 1988).

[4] B.B. Mandelbrot, J. Fluid Mech. 62 (1974) 331; also Comptes Rendus 278A (1974) 289, 355.

[5] B.B. Mandelbrot, The Fractal Geometry of Nature. (Freeman, New York, 1982).

[6] B.B. Mandelbrot, Fractals and Multifractals: Noise, Turbulence and Galaxies (Selecta, Vol. 1) (Springer, New York, forthcoming).

[7] U. Frisch and G. Parisi, in: Turbulence and Predictability in Geophysical Fluid Dynamics and Climate Dynamics, M. Ghil, ed., International School of Physics "Enrico Fermi," Course 88 (North-Holland, Amsterdam, 1985), p. 84.

[8] T.C. Halsey, M.H. Jensen, L.P. Kadanoff, I. Procaccia and B.I. Shraiman, Phys. Rev. A 33 (1986) 1141.

[9] J.D. Deutschel and D.W. Stroock, Large Deviations (Academic Press, New York, 1989).

[10] B.B. Mandelbrot, in: Fluctuations and Pattern Formation (Cargèse, 1988) H.E. Stanley and N. Ostrowsky, ed. (Kluwer, Dordrecht–Boston, 1988), p. 345.

[11] B.B. Mandelbrot, Pure Appl. Geophys. 131 (1989) 5.

[12] M.D. Bramson, Commun. Pure Appl. Math. 31 (1978) 531.

[13] B.B. Mandelbrot, in: Fractals (Proceedings of the Erice meeting, 1988) L. Pietronero, ed. (Plenum, New York, 1989).

[14] C. Meneveau and K.R. Sreenivasan, Phys. Lett. A 137 (1989) 103.

[15] A. Chhabra and R.V. Jensen, Phys. Rev. Lett. 62 (1989) 1327.

[16] M.E. Cates and J.M. Deutsch, Phys. Rev. A 35 (1987) 4907.

[17] B. Fourcade and P.A.-M.S. Tremblay, Phys. Rev. A 36 (1987) 2352.

[18] C. Meneveau and K.R. Sreenivasan, Phys. Rev. Lett. 59 (1987) 1424.

[19] R.R. Prasad, C. Meneveau and K.R. Sreenivasan, Phys. Rev. Lett. 61 (1988) 74.

[20] R. Blumenfeld and A. Aharony, Phys. Rev. Lett. 62 (1989) 2977.

[21] B.B. Mandelbrot, in: Frontiers of Physics: Landau Memorial Conference (Tel Aviv, 1988) E. Gotsman, ed. (Pergamon, New York, 1989).