SOME NOTES ON RECENT WORK OF DANI WISE

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Abstract. We summarize some of the main results and provide context and background for some recent results by Dani Wise on quasiconvex hierarchies for groups. These notes are based on a sequence of three talks given by Wise at the Wasatch Topology Conference in Park City, UT between 12/14 and 12/16/2009.

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Contents

1. Graph groups and hierarchies

Definition 1.1 (Graph group)**.** *Let Γ be a simplicial graph. The graph group of Γ, denotes $G(\Gamma)$, has the presentation*

$$\langle vertices(\Gamma) \mid [a,b] \text{ whenever } (a,b) \in edge(\Gamma)\rangle.$$

Graph groups are often known as right-angled Artin groups (RAAGs). The study of graph groups was initiated by Baudisch in [Bau], and many properties were established by Droms in [Droms1], [Droms2] and [Droms3]. Linearity was established by Green in [Gre] (cf. [Hum]). Linearity properties were also established by Hsu and Wise in [HW]. Droms established that graph groups are residually torsion-free nilpotent. They are known to be $\mathbb{Z}$-linear.

Definition 1.2 (Hierarchy, quasiconvex hierarchy)**.** *The trivial group has a length 0 hierarchy. G has a length $\leq n$ hierarchy if $G \cong A *_C B$ or $G \cong A*_{C^t=C'}$, where A and B have length $\leq (n-1)$ hierarchies. A hierarchy is called quasiconvex (resp. cyclic, finitely generated) if $C < G$ is a quasi-isometrically embedded finitely generated subgroup (resp. cyclic, finitely generated subgroup).*

Groups with merely finitely generated hierarchies can be very badly behaved. Note that we can build nontrivial groups with hierarchies by observing that the HNN extension of the trivial group over the trivial group is $\mathbb{Z}$. The main theorem of these notes is the following:

Theorem 1.3. *Let G be a word-hyperbolic group with a quasiconvex hierarchy. Then G has a finite index subgroup G' which embeds in a graph group R.*

In terminology which will be developed throughout these notes, G' is the fundamental group of a compact, special, nonpositively curved cube complex X.

Example 1.4. *The Baumslag-Solitar groups $BS(n,m)$ do not admit quasiconvex hierarchies, and neither do the Burger-Mozes simple groups (see* [BM]*).*

Recall that a one-relator group with torsion has a presentation of the form $\langle X \mid W^n \rangle$, where W is a nontrivial reduced word in X and $n \geq 2$.

Corollary 1.5. *Let G be a one-relator group with torsion. Then G is residually finite, linear, and all quasiconvex subgroups are separable.*

Corollary 1.5 was originally conjectured by Baumslag in 1968. He observed that the group

$$BS(2,3) \cong \langle b, s \mid (b^2)^s b^{-3} \rangle$$

was non-Hopfian (since $\phi : b \mapsto b^2$, $\phi : s \mapsto s$ is obviously a surjection, but $\phi([b, b^s]) = [b^2, b^3] = 1$ and $[b, b^s]$ is nontrivial in $BS(2,3)$) and hence not residually finite by a classical theorem of Mal'cev, but that it was if the relator word was replaced by a proper power.

Proof of Corollary 1.5. The proof consists of three main steps. First, every one-relator group has a hierarchy terminating with the group $\mathbb{Z}/n\mathbb{Z} * F_m$ for some n and m. This is known as the Magnus-Moldovanskii hierarchy. Secondly, every one-relator group has a torsion-free finite index subgroup. Finally, for one-relator groups with torsion, the Magnus-Moldovanskii hierarchy is quasiconvex. □

The Magnus-Moldovanskii hierarchy for a one-relator group G is obtained by replacing the generating set by a smaller set which omits at least one generator appearing in the relator word (which we assume to by cyclically reduced). A result of Newman in [NewB] shows that any subgroup appearing in the Magnus-Moldovanskii hierarchy is malnormal. Haglund and Wise develop a malnormal combination theorem to roughly obtain a proper, cocompact action on a CAT(0) cube complex.

Corollary 1.6. *Let M be a finite volume cusped hyperbolic manifold, and assume M contains a geometrically finite incompressible surface S. Then $G = \pi_1(M)$ has a finite index subgroup G' which embeds in a graph group. Equivalently, M has a finite cover $\widehat{M}$ such that $\pi_1(\widehat{M}) \cong \pi_1(X)$, where X is a compact, special, nonpositively curved cube complex.*

The claim about cusped hyperbolic manifolds requires a relative version of Theorem 1.3, as these groups might not be word hyperbolic.

A Haken hierarchy for M implies the existence of a hierarchy for $\pi_1(M)$. By the work of Thurston, this is a quasiconvex hierarchy if and only if S is geometrically finite (see [T], cf. [S]).

Corollary 1.7. *Suppose S is geometrically finite. Then $\pi_1(M)$ is subgroup separable, i.e. every finitely generated subgroup of $\pi_1(M)$ is separable.*

Proof. When X is compact and $\pi_1(X)$ is word-hyperbolic, then every quasiconvex subgroup is separable. The tameness theorem of Calegari-Gabai and Agol ([CG], [A1]), finitely generated subgroups are either geometrically finite or fundamental groups of virtual fibers. In the first case, they are quasiconvex, hence separable. In the second case, separability follows from the description of $\pi_1(M)$ as a virtual semidirect product and the LERF property of surface groups. □

Corollary 1.8. *Every Haken hyperbolic 3-manifold is virtually fibered.*

Proof. By the work of Bonahon in [B], an incompressible surface S is either a virtual fiber, or S is geometrically finite. In the first case, M virtually fibers. In the second case, there is a finite cover $\widehat{M}$ of M such that $\pi_1(\widehat{M}) < R$ for a graph group R. Each graph group is residually finite rationally solvable (RFRS). By [A2], M virtually fibers. □

Even though graph groups are residually torsion-free nilpotent, one needs another argument to show they are residually finite rationally solvable. In [K] there are examples of residually torsion-free nilpotent groups which are not RFRS, but Agol shows that graph groups are RFRS.

2. Special cube complexes

Definition 2.1 (Standard n-cube, cube complex)**.** *The standard n-cube is $[-1, 1]^n$ A subcube is obtained by restricting coordinates to ± 1. A cube complex is a complex built out of standard n-cubes by identifying subcubes.*

Definition 2.2 (Link of a 0-cube, flag complex, nonpositively curved cube complex)**.** *The link of a 0-cube is a complex of simplices whose n-simplices correspond to corners of $n+1$-cubes meeting at v, with the appropriate gluing instructions according to the adjacency relation. A flag complex is a simplicial complex such that $n+1$ vertices span an n-simplex if and only if they are pairwise adjacent. A cube complex X is nonpositively curved if the link of each vertex is a flag complex.*

Though we may think of a cube complex, all of whose vertices v satisty $\mathrm{link}(v)$ is a flag complex, as nonpositively curved by definition, this is actually a theorem of Gromov (see [G]). Often times, nonpositive curvature is stated using the CAT(0) inequality.

Example 2.3. *A square complex (i.e. a two-dimensional cube complex) is nonpositively curved if and only if for each vertex v, $\mathrm{link}(v)$ is a graph whose systole has length at least 4.*

Example 2.4. *Let R be a graph group. Then $R \cong \pi_1(X)$ for X a nonpositively curved cube complex. The 2-skeleton X^2 is the standard 2-complex of the standard presentation of R. We build X by attaching k-tori for each k-tuple of commuting generators. Many authors call this complex the Salvetti complex (see* [Sa]*, cf.* [Ch]*). This is a nonpositively curved cube complex. If Γ is a finite graph, we denote the Salvetti complex of the corresponding graph group by $X(\Gamma)$.*

Definition 2.5 (Local isometry, special cube complex)**.** *Let $\phi : Y \to X$ be a map of cube complexes. There is an induced map $\phi_* : \mathrm{link}(y) \to \mathrm{link}(x)$. We say that ϕ is a local isometry if ϕ_* is an inclusion, adjacency-preserving, and full. We say that Y is special if there is a local isometry $Y \to X(\Gamma)$ for some Γ.*

We remark that if $\phi : Y \to X$ is a local isometry, then the map on universal covers $\widetilde{\phi} : \widetilde{Y} \to \widetilde{X}$ is an isometric embedding, to that $\pi_1(Y) \to \pi_1(X)$ is quasiconvex in the compact case.

Definition 2.6 (CAT(0) cube complex)**.** *$\widetilde{X}$ is a CAT(0) cube complex if it is a simply connected nonpositively curved cube complex.*

We remark that the usual definition of CAT(0) comes from comparison geometry. A geodesic metric space X satisfies the CAT(0) inequality, and is called CAT(0), if the parametrized distance between sides of each triangle is less than the parametrized distance between the sides in a Euclidean comparison triangle. It is true that $\widetilde{X}$ admits a CAT(0) metric with each n-cube isometric to $[-1, 1]^n$.

The machine which makes CAT(0) cube complex function as desired is because of hyperplanes.

Definition 2.7 (Midcube, hyperplane)**.** *A midcube in $[-1, 1]^n$ is a subspace obtained by restricting one coordinate to 0. A hyperplane in $\widetilde{X}$ is a connected subspace intersecting each cube emptily or in a midcube.*

We remark that each midcube lies in a unique hyperplane $\widetilde{D}$. Each such hyperplane is itself a convex CAT(0) cube complex which separates $\widetilde{X}$ into two components. Immersed hyperplanes in a nonpositively curved cube complex are similarly defined.

Definition 2.8 (Special cube complex)**.** *Let X be a nonpositively curved cube complex. We say that X is special if it avoids the following four pathologies:*

1. *Self-crossing immersed hyperplanes.*
2. *One-sided hyperplanes.*
3. *Self-osculating hyperplanes.*
4. *Inter-osculating hyperplanes.*

The two definitions of special cube complexes are equivalent:

Theorem 2.9. *Y is special if and only if there is a local isometry $Y \to X(\Gamma)$. The graph Γ can be reconstructed by letting the vertices be the immersed hyperplanes of Y and the edges be given by intersections of hyperplanes.*

A general philosophy of the whole discussion is that a special cube complex should be thought of as a higher-dimensional graph. A graph is special and a CAT(0) cube complex is special.

A fundamental property of special cube complexes is the following theorem. The first statement is classical and is due to Hall, and the latter is the content.

Theorem 2.10. *Let $\phi : A \to X$ be an immersion of graphs with A compact. Then there exists a finite cover $\widetilde{X}$ such that ϕ lifts to $\widetilde{X}$ and $\widetilde{X}$ retracts onto A. The same conclusion holds when ϕ is replaced by a local isometry of special cube complexes.*

Corollary 2.11. *When X is a special and compect with $\pi_1(X)$ word hyperbolic, then every quasiconvex $H < \pi_1(X)$ is separable.*

Proof of Corollary 2.11. There exists a local isometry $\phi : A \to X$ with $\phi_* : \pi_1(A) \to H$. We have that H is a retract of $\pi_1(\widetilde{X})$. Retracts of Hausdorff topological spaces are closed in the profinite topology, whence the claim. □

To produce A, one takes a point x in $\widetilde{X}$ and considers its orbit Hx. We let A be the hull of Hx, namely the intersection of all half spaces containing Hx.

3. Cubulating malnormal amalgams

The work behind cubulating malnormal amalgams described here is due to Haglund, Hsu and Wise.

Definition 3.1 (Malnormal subgroup, almost malnormal subgroup)**.** *Let G be a group and $H < G$ a subgroup. We say that H is malnormal in G if for each $g \in G \setminus H$, we have $H \cap g^{-1}Hg = \{1\}$. We say that H is almost malnormal if $H \cap g^{-1}Hg$ is finite.*

We have the following malnormal combination theorem:

Theorem 3.2. *Suppose that $G \cong A *_C B$ or $G \cong A*_{C^t=C}$, where A, B are word hyperbolic and act properly and cocompactly on CAT(0) cube complexes. Suppose that C is almost malnormal and quasiconvex in G. Suppose that A and B satisfy certain technical extension properties which are automatically satisfied when A and B are virtually special. Then G acts properly and cocompactly on a CAT(0) cube complex.*

If $Y \subset X$ is a subcomplex, we write by $N^*(Y)$ a cubulated neighborhood of Y. The following malnormal special combination theorem can be found in [HaW]:

Theorem 3.3. *Suppose that X is a compact, nonpositively curved cube complex with $\pi_1(X)$ word hyperbolic. Suppose $D \to X$ is an embedded hyperplane and suppse that each component of $X - N^*(D)$ is virtually special. Then X is virtually special.*

Therefore, for word hyperbolic groups we see that a malnormal quasiconvex hierarchy for G gives rise to a realization of G as the fundamental group of a virtually special cube complex. This follows from combining the previous two theorems with an induction on the length of the hierarchy.

We also have the following special quotient theorem:

Theorem 3.4. *Let G be a word hyperbolic group which is the fundamental group of a virtually special compact nonpositively curved cube complex. Let $H_1, \ldots, H_k$ are quasiconvex subgroups of G. Then there exist $H_i' < H_i$ of finite index such that*

$$\overline{G} = \frac{G}{\langle\langle H_1', \ldots, H_k' \rangle\rangle}$$

is the fundamental group of a virtually special compact nonpositively curved cube complex. Furthermore, we may arrange so that $\overline{G}$ is word hyperbolic.

4. Codimension 1 subgroups and Sageev's construction

Definition 4.1 (Codimension 1 subgroup)**.** *Let H be a subgroup of a finitely generated group G with generating set S. Write Γ for the Cayley graph of G with respect to S. A subset K of Γ is deep with respect to H if for any $t > 0$ we have K is not contained in $N_t(H)$. H has codimension 1 if for some $r > 0$, $\Gamma \setminus N_r(H)$ has at least two components which are deep.*

We remark that there is an equivalent formulation of codimension 1 subgroups, which one can find in [Sag], for instance. If G is finitely generated and $H < G$, we can consider the relative ends of G, $e(G, H)$. This is simply the number of ends of the coset graph of H. H has codimension 1 if $e(G, H) > 1$.

Example 4.2. *Any copy of $\mathbb{Z}^n$ in $\mathbb{Z}^{n+1}$ has codimension 1. Essential immersed circles in surfaces give rise to codimension 1 subgroups, as do incompressible immersed surfaces in 3-manifolds.*

Sageev's construction, as developed in [Sag], is as follows: we let $H_1, H_2, \ldots, H_k$ be codimension 1 subgroups of G. Then G acts on a CAT(0) cube complex $\widetilde{X}$, and this complex is dual to a certain associated wall space. The stabilizers of hyperplanes of $\widetilde{X}$ are all commensurable with conjugates of the H_i. When G and the H_is are nice, then G acts properly, cocompactly, and $\widetilde{X}$ has finite dimension. For example, Sageev proves these conclusions when G is hyperbolic and the H_i are quasiconvex. A generalization of this result to relatively hyperbolic groups has been carried out by Hruska and Wise.

5. Propaganda

In this section we introduce Wise's grand scheme for understanding groups.

(1) Find codimension 1 subgroups: for example, when we can write $G = A *_C B$ with C nontrivial. Then all the subgroups A, B, C are codimension 1. Call these subgroups $H_1, \ldots, H_n$.
(2) Cubulate: obtain a G-action on a CAT(0) cube complex $\widetilde{X}$.
(3) Recognize certain finiteness properties of $\widetilde{X}$ from codimension 1 subgroups: properness and cocompactness are particularly desirable.
(4) Attempt to prove separability properties, i.e. closure in the profinite topology, of the H_i or $H_i g H_j$ for $g \in G$.

(5) Obtain virtual specialness, i.e. $J < G$ of finite index such that $\widetilde{X}/J$ is special.
(6) J has a relatively simple, understandable structure.

One should apply this scheme when G is a 3-manifold groups, one-relator groups, HNN extensions of free groups by cyclic subgroups, matrix groups, etc., to obtain and study various properties which may not be algebraically available. Namely, one obtains information about these groups which are not direct consequences of their descriptions as 3-manifold groups, one-relator groups, etc. After cubulating, one obtains a nonpositively curved cube complex whose dimension may be enormous and divorced from any natural invariants associated to G. For example, the HNN extension

$$\langle a, b, t \mid [a, b]^t = [a^{10}, b^{10}]\rangle$$

requires something on the order of 100 dimensions to cubulate.

6. Small cancellation theory

We begin by recalling some basic results concerning small cancellation theory. Most of this material can be found in [LS], for instance. Let $G = \langle X \mid R\rangle$ be a group presentation. We assume that the set R consists of freely and cyclically reduced words, and we assume that R is closed under taking cyclic permutations and inverses. Such an R is called a **symmetrized set of relations** for the presentation. Clearly we may modify any presentation for G to be symmetrized.

Definition 6.1 (Piece, small cancellation condition)**.** *Let u be a nontrivial freely reduced word in $F(X)$. We say that u is a piece with respect to a presentation if u is an initial segment of two distinct words in R. Let $0 < \lambda < 1$. The presentation of G satisfies the $C'(\lambda)$ small cancellation condition, also known as the metric small cancellation condition, if whenever u is a piece of some $r \in R$ we have that $\ell(u) < \lambda\ell(r)$. If p is a number greater than 2, we say that the presentation satisfies the $C(p)$ small cancellation condition, also known as the non-metric cancellation condition, if whenever $r \in R$ is a product of m pieces then $p \leq m$. Let $q > 2$. The presentation satisfies the $T(q)$ non-metric small cancellation condition if whenever $3 \leq t < q$ and $r_1, \ldots, r_t$ are elements of R such that $r_i \neq r_{i+1}^{-1}$ where the indices are considered modulo t, then at least one of the products r_ir_{i+1} is freely reduced.*

Notions in small cancellation theory are often understood in terms of Van Kampen diagrams, which are convenient geometric tools for studying the word problem in a group. For the convenience of the reader, we will recall some of the basic theory of Van Kampen diagrams here.

Definition 6.2 (Van Kampen diagram)**.** *Let G be presented as above. A Van Kampen diagram over the presentation is a planar, finite cell complex $\mathcal{D}$ equipped with a piecewise-linear embedding in $\mathbb{R}^2$, and it satisfies the following four conditions:*

(1) *$\mathcal{D}$ is simply connected. In particular it is connected.*
(2) *Each edge of $\mathcal{D}$ is labelled by a direction and a letter in X. Traversing an edge in a particular direction corresponds to multiplying by a generator or its inverse.*

(3) *There is a distinguished vertex O on $\partial\mathcal{D} \subset \mathbb{R}^2$, the basepoint.*
(4) *For each region of $\mathcal{D}$ and each vertex on the boundary, the boundary cycle (in either direction) corresponding to the region is labelled (starting at the preordained vertex) with a freely reduced word in R.*

We say that a Van Kampen diagram is reduced if there is no reduction pair. Namely, there is no pair of distinct regions whose boundary cycles share a common edge and such that their boundary cycles, starting at the shared edge, have a common initial segment. We think of such a diagram to be foldable, and we reduce it by folding.

Lemma 6.3 (Van Kampen's Lemma)**.** *If $\mathcal{D}$ is a Van Kampen diagram over a presentation with boundary label w which is not necessarily freely reduced, then w represents the identity in G. If the area of $\mathcal{D}$ is no more than n, then w is a product of no more than n conjugates of elements of R. Conversely, if w represents the identity in G then there exists a reduced Van Kampen diagram whose boundary label is freely reduced and equal to w. If w is a product of n conjugates of elements of R then $\mathcal{D}$ can be taken to have area no more than n.*

In the context of Van Kampen diagrams, $C(p)$ and $T(q)$ have some nice and easy consequences. If a presentation satisfies $C(p)$ then each region of $\mathcal{D}$ whose boundary does not meet $\partial\mathcal{D}$ in an edge has edge length at least p. If a presentation satisfies $T(q)$ then each interior vertex of $\mathcal{D}$ has degree at least q. The first of these claims is obvious since each interior edge of $\mathcal{D}$ is labelled by a piece. The other claim is equally easy: let v be such a vertex with emanating edges labelled $f_1, \ldots, f_h$. Then there are paths $c_1, \ldots, c_h$ such that $r_i = f_i^{-1} c_i f_{i+1}$. Since there can no cancellation in some $r_i r_{i+1}$, $T(q)$ fails if $q > h$.

Example 6.4. *Let G be the fundamental group of a surface of genus g whose presentation is the symmetrization of the presentation*

$$G = \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid \prod_{i=1}^{g} [a_i, b_i] \rangle.$$

Then the only pieces are words of length 1. So, the presentation satisfies $C(4g)$ and $C'(1/(4g-1))$.

The fundamental result in small cancellation theory is Greendlinger's Lemma:

Theorem 6.5. *Suppose that a group presentation for G satisfies $C'(\lambda)$ for $0 \leq \lambda \leq 1/6)$, and $w \in F(X)$ a nontrivial freely reduced word which reduces to the identity in G. Then there is a subword v of w and $r \in R$ such that v is a subword of r and such that*

$$\ell(v) > (1 - 3\lambda)\ell(r).$$

A proof of Greendlinger's Lemma can be provided by developing an appropriate notion of combinatorial curvature. Small cancellation groups have many nice properties. The following properties are enjoyed by $C'(1/6)$ groups:

- G is torsion-free if and only if no relator is conjugate to a proper power.
- G is word-hyperbolic.
- G has a solvable word and conjugacy problem.
- Any two equivalent symmetrized finite relator sets are equal.

There are many generalizations of small cancellation theory, and the one developed by Wise is the last ingredient in his proof of Theorem 1.3.

Definition 6.6 (Cubical relative presentation)**.** *A cubical relative presentation is a collection of the data $\langle X \mid Y_1, \ldots, Y_r\rangle$, where X is a nonpositively curved cube complex and each Y_i is equipped with a local isometry to X. We attach a mapping cone along each Y_i to obtain a complex X^*, whose fundamental group is the cubical relative presentation.*

Cubical relative presentations have a small cancellation theory whose pieces are overlaps between the relators and hyperplanes. The theory arises when the pieces in each Y_i are small compared to the systole in Y_i.

In usual small cancellation theory, a presentation of the form

$$\langle X \mid w_1^{n_1}, \ldots, w_k^{n_k}\rangle,$$

where no w_i is conjugate to w_j for $i \neq j$ has a small cancellation theory if the exponents are chosen to be sufficiently large. Analogously, Wise has:

Theorem 6.7. *Let X be a nonpositively curved cube complex and $Y_i \to X$ a compact local isometry, $1 \leq i \leq r$, $\pi_1(Y_i)$ malnormal for all i, and $\pi_1(Y_i)$ and $\pi_1(Y_j)$ not sharing any nontrivial conjugacy classes. Then $\langle X \mid \widehat{Y}_i\rangle$ has a small cancellation theory as a cubical relative presentation for some finite covers $\widehat{Y}_i \to Y_i$ of sufficiently large girth, in some technical sense. If X is compact and each immersed hyperplane has a separable fundamental group, then it is possible to arrange so that $\pi_1(X^*)$ has a quasiconvex hierarchy.*

Theorem 6.8 (Special quotient theorem)**.** *Let $G = \pi_1(X)$ be word hyperbolic with X is virtually special and compact. If $H_1, \ldots, H_k$ are quasiconvex, there there are finite index subgroups $H_i' < H_i$ such that $G/\langle\langle H_1', \ldots, H_k'\rangle\rangle$ is virtually special.*

7. Applications to the author's work

In this section we will prove the following theorem, which appears in [K] and answers positively a question posed to the author by Lubotzky:

Theorem 7.1. *Let $\Gamma < PSL_2(\mathbb{C})$ be a lattice and let p be a prime. Then Γ is virtually residually finite p.*

Any such Γ is the fundamental group of a hyperbolic orbifold. After passing to a finite cover, we may assume that Γ is the fundamental group of a hyperbolic 3-manifold M of finite volume. By a result due to Thurston, the representation $\pi_1(M) \to PSL_2(\mathbb{C})$ lifts to $SL_2(\mathbb{C})$ (cf. [CS]).

Let $\mathcal{R} = \mathcal{R}(\Gamma)$ denote the $SL_2(\mathbb{C})$ representation variety of Γ. By general theory (see [R], for instance), $\mathcal{R}$ contains a point over $\overline{\mathbb{Q}}$ and in fact a faithful representation $\Gamma \to SL_2(\overline{\mathbb{Q}})$. Since Γ is finitely generated, there is a finite extension $K/\mathbb{Q}$ such that the image of Γ lands in $SL_2(K)$. We let $\mathcal{O}$ denote the ring of integers in K. In any matrix in the image, there are at most four denominators, and so any finite generating set for Γ has only finitely many denominators occurring among nonzero entries in the image of Γ. Fix a finite generating set for Γ and consider the denominators which occur. These will be contained in finitely many prime ideals in $\mathcal{O}$. Each prime ideal of $\mathcal{O}$ lies over a unique prime ideal $p\mathbb{Z}$. For the set of denominators which occur in the image a generating set for Γ, let $\mathcal{B} \subset \mathbb{Z}$ be the finite set of primes over which the associated prime ideals in $\mathcal{O}$ lie. We call $\mathcal{B}$ the set of bad primes.

Lemma 7.2. *Let p be any prime. When $\mathcal{B}$ is empty, Γ is virtually residually p.*

Proof. This is entirely analogous to the fact that $SL_2(\mathbb{Z})$ is virtually residually p and is done using the first congruence subgroup. Let P lie over $p\mathbb{Z}$. Let Γ_1 denote the kernel of the natural map

$$SL_2(\mathcal{O}) \to SL_2(\mathcal{O}/P).$$

We have a natural action of Γ_1 on $\mathcal{O}^2$, and we can construct the semidirect product

$$1 \to \mathcal{O}^2 \to G \to \Gamma_1 \to 1.$$

We can also construct truncated semidirect products of the form

$$1 \to (\mathcal{O}/P^n)^2 \to G_n \to \Gamma_1 \to 1.$$

By considering the successive quotients $(P^i/P^{i+1})^2$, we see that the conjugation action of Γ_1 on $(\mathcal{O}/P^n)^2$ is unipotent. Let $K_n < \Gamma_1$ denote the kernel of this action. We have that $\mathcal{O}/P^n$ is always a p-group. It follows that the semidirect product

$$1 \to (\mathcal{O}/P^n)^2 \to \overline{G}_n \to \Gamma_1/K_n \to 1$$

is a p-group, so that

$$\bigcap_n K_n = \{1\}$$

and Γ_1/K_n is a p-group for all n (a similar argument is fleshed out in [K]). $\square$

Alternatively, the argument could have proceeded as follows: $(\mathcal{O}/P^n)^2$ is a p-group, and $(P/P^n)^2$ is its Frattini subgroup. On the other hand, it is classical that if Q is a p-group and $\phi(Q)$ is its Frattini subgroup, then the group of automorphisms of Q which induce the identity on $Q/\phi(Q)$ form a p-group, whence the conclusion.

We now consider the case where $\mathcal{B} \neq \emptyset$. Fix $P \in \mathcal{B}$, and let $\widehat{\mathcal{O}} = \widehat{\mathcal{O}}_P$ be the completion of $\mathcal{O}$ at P, namely

$$\widehat{\mathcal{O}} = \varprojlim \mathcal{O}/P^n.$$

We let $\widehat{K}$ be the fraction field of $\widehat{\mathcal{O}}$. We have a canonical map $\mathcal{O} \to \widehat{\mathcal{O}}$ which is injective since $\mathcal{O}$ is a Dedekind domain, and thus we have an injective map $K \to \widehat{K}$, and a faithful representation $\Gamma \to SL_2(\widehat{K})$ induced by the inclusion $SL_2(K) \to SL_2(\widehat{K})$. By construction, the field $\widehat{K}$ comes equipped with a discrete valuation ν. Explicitly, it takes an equivalence class of fractions

$\gamma = \alpha/\beta$, determines an i and j such that $\alpha \in P_i \setminus P^{i+1}$ and $\beta \in P^j \setminus P^{j+1}$ and sets $\nu(\gamma) = i - j$. By abuse of notation, we write P as the maximal ideal generated by the image of P in $\widehat{\mathcal{O}}$. The valuation ν thus defined is a discrete valuation, so that $\widehat{\mathcal{O}}$ is a DVR.

Let V be a two-dimensional vector space over $\widehat{K}$. Recall that at $\widehat{\mathcal{O}}$-lattice in V is a rank 2 $\widehat{\mathcal{O}}$-module which spans V as a $\widehat{K}$-vector space. Let L be a $\widehat{\mathcal{O}}$-lattice and L' a sublattice. Then L/L' is isomorphic to $\widehat{\mathcal{O}}/P^a \oplus \widehat{\mathcal{O}}/P^b$ for some choice of nonnegative integers a and B. There is a natural action of $\widehat{K}$ on the set of $\widehat{\mathcal{O}}$-lattices in V. Note that if L and L' are arbitrary lattices, we can replace L' by an equivalent lattice kL' such that $kL' \subset L$ by choosing an appropriate $k \in \widehat{K}$. We break the $\widehat{K}$-orbits into equivalence classes. There is a natural graph whose vertices are equivalence classes of lattices, and whose edges span pairs of equivalence classes for which there exist representatives satisfying $L/L' \cong \widehat{\mathcal{O}}/P$. It is shown in [Se] that this graph is a tree, called the lattice tree of $\widehat{\mathcal{O}}$.

We have that $SL_2(\widehat{K})$ acts on this tree in the obvious way. The stabilizers of vertices are precisely the $GL_2(\widehat{K})$-conjugates of $SL_2(\widehat{\mathcal{O}})$. The following lemma follows easily from this discussion.

Lemma 7.3. *If Γ is as above, then either Γ is virtually residually p or Γ acts on the lattice tree of $\widehat{\mathcal{O}}$ without a global fixed point.*

General Bass-Serre theory (cf. [Se]) therefore implies that when Γ acts nontrivially on the lattice tree, then Γ splits as a nontrivial amalgamated product. Furthermore, the amalgamating group can, up to conjugacy be taken to be the image of Γ in $SL_2(\widehat{\mathcal{O}})$. Since Γ is the fundamental group of a hyperbolic manifold, the amalgamating group is nontrivial. Indeed, $\mathbb{H}^3/\Gamma$ is irreducible and hence Γ cannot split as a nontrivial free product.

The final ingredient we need is the following, which is due to Epstein, Stallings and Waldhausen, and a proof can be found in [CS].

Lemma 7.4. *Let M be a compact, orientable 3-manifold. For any nontrivial splitting of $\pi_1(M)$ there exists a nonempty system S of incompressible non-peripheral surfaces such that the image of the inclusion on fundamental groups is contained in an edge group. Furthermore, the image of the fundamental groups of the components of $M \setminus S$ are contained in a vertex group.*

This lemma applies to our situation, since by [A1] and [CG], $\mathbb{H}^3/\Gamma$ is homeomorphic to the interior of a compact 3-manifold. From here, Theorem 7.1 is obvious:

Proof of Theorem 7.1. If Γ cannot be coaxed into admitting a faithful representation into $SL_2(\mathcal{O})$, we have that $\mathcal{B} \neq \emptyset$. For each $p \in \mathcal{B}$, choose $P \subset \mathcal{O}$ lying over p. We constructed a faithful representation of Γ into $SL_2(\widehat{K})$ whose image does not lie in $SL_2(\widehat{\mathcal{O}}_P)$. But then we obtain a nontrivial splitting of Γ, and conclude that $\mathbb{H}^3/\Gamma$ is Haken. By a corollary to Theorem 1.3, it follows that $\mathbb{H}^3/\Gamma$ virtually fibers. It follows from the work in [K] that any 3-manifold which virtually fibers over the circle has a virtually residually p fundamental group. □

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