Right-angled Artin groups in the C^∞ diffeomorphism group of the real line

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ABSTRACT. We prove that every right-angled Artin group embeds into the C^∞ diffeomorphism group of the real line. As a corollary, we show every limit group, and more generally every countable residually RAAG group, embeds into the C^∞ diffeomorphism group of the real line.

1. Introduction

The *right-angled Artin group* on a finite simplicial graph Γ is the following group presentation:

$$A(\Gamma) = \langle V(\Gamma) \mid [v_i, v_j] = 1 \text{ if and only if } \{v_i, v_j\} \in E(\Gamma)\rangle.$$

Here, $V(\Gamma)$ and $E(\Gamma)$ denote the vertex set and the edge set of Γ, respectively.

For a smooth oriented manifold X, we let $\mathrm{Diff}_+^\infty(X)$ denote the group of orientation preserving C^∞ diffeomorphisms on X. A group G is said to *embed* into another group H if there is an injective group homormophism $G \to H$. Our main result is the following.

Theorem 1. *Every right-angled Artin group embeds into* $\mathrm{Diff}_+^\infty(\mathbb{R})$.

Recall that a finitely generated group G is a *limit group* (or a *fully residually free group*) if for each finite set $F \subset G$, there exists a homomorphism ϕ_F from G to a nonabelian free group such that ϕ_F is an injection when restricted to F. The class of limit groups fits into a larger class of groups, which we call *residually RAAG groups*. A group G is in this class if for each $1 \neq g \in G$, there exists a graph Γ_g and a homomorphism $\phi_g \colon G \to A(\Gamma_g)$ such that $\phi_g(g) \neq 1$. Since the class of right-angled Artin groups is closed under taking finite direct products, we could replace g with an arbitrary finite subset of G. Theorem 1 can be strengthened as follows.

Corollary 2. *Every countable residually RAAG group embeds into* $\mathrm{Diff}_+^\infty(\mathbb{R})$.

A similar argument to the proof of Theorem 1 also applies to the group $\mathrm{PL}_+(\mathbb{R})$ of orientation preserving piecewise-linear homeomorphisms of $\mathbb{R}$.

Corollary 3. *Every countable residually RAAG group embeds into* $\mathrm{PL}_+(\mathbb{R})$.

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Our construction requires infinitely many domains of linearity in $\mathbb{R}$, so that we must take infinite subdivisions of $\mathbb{R}$ in order to get residually RAAG groups inside of $\mathrm{PL}_+(\mathbb{R})$. The reader may compare Corollary 3 with the work of Brin and Squier [6], which shows that the group $\mathrm{PLF}(\mathbb{R})$ of piecewise-linear homeomorphisms of $\mathbb{R}$ with finite subdivisions contains no nonabelian free subgroup.

Recall that a group G is called *virtually special* if a finite index subgroup $H \leq G$ acts properly and cocompactly on a CAT(0) cube complex X such that the quotient $H\backslash X$ avoids certain pathologies of its half–planes (see [16]). A consequence of such an action is that the subgroup H embeds in a right-angled Artin group.

Corollary 4. *Let G be a group which is virtually special. Then there is a finite index subgroup $H \leq G$ which embeds into $\mathrm{Diff}^{\infty}_{+}(\mathbb{R})$ and also into $\mathrm{PL}_+(\mathbb{R})$.*

Examples of virtually special groups include fundamental groups of closed surfaces and finite volume hyperbolic 3–manifold groups [1, 22]. Combining the work of Bergeron–Haglund–Wise [3] and Bergeron–Wise [4], we have that there are virtually special closed hyperbolic manifolds in all dimensions.

The finitely presented subgroups of diffeomorphism groups are generally very complicated. In [5], Bridson used a virtually special version of the Rips machine to produce finitely presented subgroups of right-angled Artin groups with exotic algorithmic properties. An arbitrary class of groups which contains sufficiently complicated right-angled Artin groups thereby also has finitely presented subgroups with exotic algorithmic properties (cf. [18]):

Corollary 5. *Suppose $G = \mathrm{Diff}^{\infty}_{+}(\mathbb{R})$ or $G = \mathrm{PL}_+(\mathbb{R})$. Then there is a finitely presented subgroup $H \leq G$ such that the conjugacy problem in H is unsolvable. Furthermore, the isomorphism problem for the class of finitely presented subgroups of G is unsolvable.*

The reader may contrast Corollary 5 with Thompson's groups F, T, and V, in which the conjugacy problem is generally solvable [2]. The group T can be embedded into $\mathrm{Diff}^{\infty}_{+}(\mathbb{S}^1)$ [15].

The group H in Corollary 5 is not conjugacy separable, since conjugacy separable groups have solvable conjugacy problems. In [13], Farb and Franks showed that the group of real analytic diffeomorphisms of $\mathbb{R}$ contains non–solvable Baumslag–Solitar groups, which are not even residually finite.

1.1. **Notes and references.** It is well-known that right-angled Artin groups embed in $\mathrm{Homeo}(\mathbb{R})$, as follows from the fact that right-angled Artin groups admit left–invariant orders. Similarly, right-angled Artin groups embed in $\mathrm{Homeo}(\mathbb{S}^1)$ because they admit left–invariant cyclic orderings. Alternatively, one can embed an arbitrary right-angled Artin group into the mapping class group $\mathrm{Mod}(S)$ of a compact surface with one boundary, and then embed $\mathrm{Mod}(S)$ into $\mathrm{Homeo}(\mathbb{S}^1)$. As noted in [12, p.47], it is generally difficult to smoothen these embeddings. The reader may consult [20] for a general discussion of the relationship between linear and cyclic orderings of groups and embeddings into $\mathrm{Homeo}(\mathbb{R})$ and $\mathrm{Homeo}(\mathbb{S}^1)$.

Farb and Franks [14] proved that every residually torsion–free nilpotent group embeds in the group of C^1 diffeomorphisms of the interval and of the circle. This implies that residually RAAG groups embed in the group of C^1 diffeomorphisms of both the interval and of the circle. Their construction does not allow for twice differentiable diffeomorphisms. In fact, Plante and Thurston [21] showed that nilpotent groups of C^2 diffeomorphisms of the interval or of the circle are abelian. In the same vein, Farb and Franks [14] show that every nilpotent subgroup of C^2 diffeomorphisms of $\mathbb{R}$ is metabelian.

For the case when the dimension is two, Calegari and Rolfsen proved that every right-angled Artin group embeds into the piecewise linear homeomorphism group of a square fixing the boundary [7]. M. Kapovich showed that every right-angled Artin group embeds into the symplectomorphism group of the sphere [17]. The second and the third named authors refined this result by embedding every right-angled Artin group into the symplectomorphism groups of the disk and of the sphere by quasi-isometric group embeddings [18].

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3. Building an injective homomorphism

Throughout this section, we let Γ be a finite graph. Consider an element g of $A(\Gamma)$. A *reduced word representing* g means a minimal length word in the standard generating set $V(\Gamma) \cup V(\Gamma)^{-1}$ representing g. The *support of* g is the set of vertices v of Γ such that v or v^{-1} appears in a reduced word representing g. We denote the support of g by $\mathrm{supp}(g)$. We say g is a *clique word* if every pair of vertices in $\mathrm{supp}(g)$ are adjacent in Γ. A *clique word decomposition for* g is the concatenation $w_k \cdots w_1$ of clique words $w_1, w_2, \ldots, w_k$ such that the concatenation is still reduced and represents g in $A(\Gamma)$. Since vertices themselves are clique words, every element in $A(\Gamma)$ has a clique word decomposition. A clique word decomposition $w_k \cdots w_1$ is *left–greedy* if for each $i < k$ and for each $v \in \mathrm{supp}(w_i)$, there exists a vertex $v' \in \mathrm{supp}(w_{i+1})$ such that $[v, v'] \neq 1$. The left–greedy clique word decomposition can be compared to the left–greedy normal form used in [19].

Lemma 6. *Every element of $A(\Gamma)$ admits a left–greedy clique word decomposition.*

Proof. Fix an element $g \in A(\Gamma)$. Let us define the *complexity* of a clique word decomposition $w_k w_{k-1} \cdots w_1$ for g as the k-tuple consisting of the word lengths $(|w_k|, |w_{k-1}|, \ldots, |w_1|)$. In the lexicographical order, this complexity is bounded above by $(|g|)$. Let us assume $w_k w_{k-1} \cdots w_1$ is the maximal clique word decomposition for g in this order. If $w_k \cdots w_1$ is not left–greedy, then there exists i and $v \in \operatorname{supp}(w_{i-1})$ be such that $v \cup \operatorname{supp}(w_i)$ spans a complete subgraph of Γ. Then we can move an occurrence of v or v^{-1} in w_{i-1} to w_i, i.e. we *slide to the left.* This is a contradiction to the maximality. □

Let X be a smooth oriented manifold. If $f \in \operatorname{Diff}_+^\infty(X)$, then we write the fixed point set of f as $\operatorname{Fix}(f)$. The closure of $X \setminus \operatorname{Fix}(f)$ will be denoted as $\operatorname{supp}(f)$. For $f_1, f_2, \ldots, f_n \in \operatorname{Diff}_+^\infty(X)$, let us define the *disjointness graph* Λ by $V(\Lambda) = \{v_1, v_2, \ldots, v_n\}$ and

$$E(\Lambda) = \{\{v_i, v_j\} \colon i \neq j \text{ and } \operatorname{supp}(f_i) \cap \operatorname{supp}(f_j) = \varnothing\}.$$

Note that two self-maps with disjoint supports commute. Hence, we have a group homomorphism $\phi \colon A(\Lambda) \to \operatorname{Diff}_+^\infty(X)$ satisfying $\phi(v_i) = f_i$. Often, it is not an obvious task at all to decide whether or not such a map ϕ is injective; see [11, 17, 18, 7] for related work on diffeomorphism groups and [9, 10, 19, 8] on mapping class groups.

Let H be a group. Another group G is *residually* H if for each nontrivial element g in G there exists a group homomorphism $\phi_g \colon G \to H$ such that $\phi_g(g) \neq 1$. For a smooth oriented manifold X, we let $\operatorname{Diff}_+^\infty(X, \partial X)$ denote the group of orientation preserving C^∞ diffeomorphisms on X which restrict to the identity near ∂X. The crucial fact we need is the following:

Lemma 7. *The group $A(\Gamma)$ is residually* $\operatorname{Diff}_+^\infty(I, \partial I)$.

Proof. Let us fix an element g in $A(\Gamma)$. We let $w_k \cdots w_1$ be a left–greedy clique decomposition representing g. We will construct a group homomorphism

$$\phi_g \colon A(\Gamma) \to \operatorname{Diff}_+^\infty(I, \partial I)$$

such that $\phi_g(g) \neq 1$. We can inductively choose a (possibly redundant) sequence

$$v_1 \in \operatorname{supp}(w_1), v_2 \in \operatorname{supp}(w_2), \ldots, v_k \in \operatorname{supp}(w_k)$$

such that $[v_i, v_{i+1}] \neq 1$ for each $i = 1, 2, \ldots, k-1$. There exists $\sigma_i \in \{-1, 1\}$ and $n_i > 0$ such that $v_i^{\sigma_i n_i}$ is the highest power of v_i in w_i. This means that $v_i \notin \operatorname{supp}(w_i v_i^{-\sigma_i n_i})$.

We choose $\rho \in \operatorname{Diff}_+^\infty(\mathbb{R})$ such that $\rho(1/4) = 5/4$ and $\rho(x) = x$ for $x \leq 0$ or $x \geq 3/2$. Put $\rho_i(x) = \rho(x-i)+i$ and $I_i = [i, i+3/2]$, so that $\operatorname{supp} \rho_i \subseteq I_i$. Note that $I_i \cap I_j = \emptyset$ for $|i - j| > 1$.

For each $v \in V(\Gamma)$, we define

$$\psi_g(v) = \prod_{v_j = v} \rho_j^{\sigma_j}.$$

This means that if $\{j\colon v_j = v\} = \{j_1 < j_2 < \cdots < j_n\}$ then $\psi_g(v) = \rho_{j_1}^{\sigma_1}\rho_{j_2}^{\sigma_2}\cdots\rho_{j_n}^{\sigma_n}$. We use the convention that the empty multiplication is trivial. Let us write $J_v = \cup_{v_j=v} I_j \supseteq \operatorname{supp}\psi_g(v)$. Note that if $v_i = v = v_j$ and $i \neq j$, then $I_i \cap I_j = \emptyset$. For each $\{u,v\} \in E(\Gamma)$, the choice of $v_1, v_2, \ldots, v_k$ implies that $J_u \cap J_v = \emptyset$. In other words, Γ is a subgraph of the disjointness graph of $\{\psi_g(v)\colon v \in V(\Gamma)\}$. It follows that ψ_g defines a group homomorphism from $A(\Gamma)$ to $\mathrm{Diff}^\infty_+(\mathbb{R})$.

Suppose $\ell \in \{1,2,\ldots,k\}$. For every $u \in \operatorname{supp} w_\ell \setminus \{v_\ell\}$, we have $J_u \cap J_{v_\ell} = \emptyset$. Since $\ell + 1/4 \in I_\ell \subseteq J_{v_\ell}$, we see that

$$\psi_g(w_\ell)(\ell+1/4) = \psi_g(v_\ell^{\sigma_\ell n_\ell})(\ell+1/4) = \rho_\ell^{n_\ell}(\ell+1/4) \geq \ell + 5/4.$$

It follows that $\psi_g(g)(5/4) = \psi_g(w_k w_{k-1}\cdots w_1)(1+1/4) \geq k + 5/4$. In particular, $\psi_g(g)$ is not the identity. By restricting the image of ψ_g onto the interval $[0, k+2]$ and conjugating by a diffeomorphism $[0,k+2] \approx I$, we obtain a desired group homomorphism ϕ_g. □

We have the following general fact.

Lemma 8. *Let G and H be groups. If G is countable and residually H, then G embeds into the countable direct product $\prod_{\mathbb{Z}} H$.*

Proof. Define

$$\psi\colon G \to \prod_{g\in G} H$$

by $\psi(x)(g) = \phi_g(x)$, where ϕ_g is as in the definition of residually H group. □

Since $\mathrm{Diff}^\infty_+([0,\infty),\{0\}) \hookrightarrow \mathrm{Diff}^\infty_+(\mathbb{R})$, Theorem 1 a trivial consequence of the following:

Theorem 9. *Every right-angled Artin group embeds into $\mathrm{Diff}^\infty_+([0,\infty),\{0\})$.*

Proof. Immediate from Lemmas 7 and 8, as well as the fact that

$$\prod_{\mathbb{Z}} \mathrm{Diff}^\infty_+(I,\partial I) \hookrightarrow \mathrm{Diff}^\infty_+([0,\infty),\{0\}).$$ □

Proof of Corollary 2. Lemma 7 implies that a residually RAAG is residually $\mathrm{Diff}^\infty_+(I,\partial I)$. We proceed as Theorem 9. □

Proof of Corollary 3. It suffices to show that every right-angled Artin group is residually $\mathrm{PL}_+(I)$. For this, we follow the proof of Lemma 7 by using $\rho_0 \in \mathrm{PL}_+(\mathbb{R})$ defined by

$$\rho_0(x) = \begin{cases} x & \text{if } x < 0 \text{ or } x \geq \frac{3}{2} \\ 5x & \text{if } 0 \leq x < \frac{1}{4} \\ (x+6)/5 & \text{if } \frac{1}{4} \leq x < \frac{3}{2} \end{cases}$$

instead of ρ. □

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