ASYMPTOTIC LINEARITY OF THE MAPPING CLASS GROUP AND A HOMOLOGICAL VERSION OF THE NIELSEN–THURSTON CLASSIFICATION

THOMAS KOBERDA

Abstract. We study the action of the mapping class group on the integral homology of finite covers of a topological surface. We use the homological representation of the mapping class to construct a faithful infinite–dimensional representation of the mapping class group. We show that this representation detects the Nielsen–Thurston classification of each mapping class. We then discuss some examples that occur in the theory of braid groups and develop an analogous theory for automorphisms of free groups. We close with some open problems.

January 20, 2011

Contents

1. Introduction, motivation and statement of results

The primary subjects of this paper are surfaces and their mapping class groups. The **mapping class group** of a surface Σ is defined to be the group of self–homeomorphisms up to isotopy, and is usually denoted $\mathrm{Mod}(\Sigma)$. A major unsolved problem in the theory of mapping class groups is whether or not they are linear (see Farb's article in [F], for instance). For some recent results towards resolving this problem for mapping class groups, braid

1991 *Mathematics Subject Classification.* Primary 57M10; Secondary 57M99.
Key words and phrases. Mapping class groups, braid groups, Nielsen–Thurston classification, representations of the mapping class group, automorphisms of free groups.

groups, and automorphism groups of free groups, the reader might consult [B1], [BiBu] and [FP].

It is natural to look at representations of the mapping class group, especially ones which arise in topological and geometric contexts, and analyze their faithfulness. One obvious candidate is the representation of $\mathrm{Mod}(\Sigma)$ as a group of automorphisms of $H_1(\Sigma, \mathbb{Z})$. We obtain a representation

$$\mathrm{Mod}(\Sigma) \to Sp_{2g}(\mathbb{Z})$$

when Σ is closed of genus g, and otherwise

$$\mathrm{Mod}(\Sigma) \to GL_n(\mathbb{Z})$$

where $n = \mathrm{rk}\, H_1(\Sigma, \mathbb{Z})$. This representation is given by taking a mapping class ψ, lifting it to a homeomorphism of Σ, and then looking at its action on the homology of Σ. This is called the **homology representation** of the mapping class group.

The homology representation of the mapping class group carries a great deal of information about mapping classes but also forgets a lot of information. The goal of this paper is to show that if one is willing to consider the actions of mapping classes on the homology of finite covers of Σ, one can recover much of the information that the homological representation forgets. This is where the "asymptotic" part of the title comes from: we see more and more nontrivial mapping classes acting nontrivially on the homology of covers as we look at higher and higher covers of Σ. Asymptotic properties of mapping class groups, especially asymptotic linearity, have been studied by many authors: see Andersen's work in [A], for example. Before we state the main results, we fix some notation.

Let $\Sigma = \Sigma_{g,n}$ be a connected surface of genus g and n punctures where $g+n \geq 3$. All surfaces we consider will be orientable. Fix a basepoint $* \in \Sigma^0$, the interior of Σ. Throughout we will denote the fundamental group of Σ by $\pi_1(\Sigma, *)$, though we will suppress the basepoint in the notation. The marked mapping class group of Σ is defined as

$$\mathrm{Mod}^1(\Sigma) = \pi_0(\mathrm{Homeo}^+(\Sigma), *).$$

We do not necessarily puncture the surface at the marked point. We identify homotopy classes of curves based at $*$ with elements of $\pi_1(\Sigma, *)$, and we identify the free homotopy classes of essential closed curves in Σ with conjugacy classes in $\pi_1(\Sigma)$. We call a homotopy class in $\pi_1(\Sigma)$ **nonperipheral** if it is not freely homotopic to a curve contained in a small neighborhood of a puncture.

We identify $\mathrm{Mod}^1(\Sigma)$ with a subgroup of $\mathrm{Aut}^+(\pi_1(\Sigma))$ and $\mathrm{Mod}(\Sigma)$ with a subgroup of $\mathrm{Out}^+(\pi_1(\Sigma))$ by simply looking at the action of homeomorphisms on based or unbased homotopy classes of curves on Σ. We perform this identification once and for all, so that the lifts of mapping classes to covers are unambiguous as far as their actions on homology are concerned. It is a standard fact that when Σ is closed, then $\mathrm{Mod}(\Sigma)$ is equal to $\mathrm{Out}^+(\pi_1(\Sigma))$, the group of orientation preserving outer automorphisms of the fundamental group of Σ. When Σ is not closed then homeomorphisms must preserve the conjugacy classes of homotopy classes of loops which encircle boundary components or punctures.

A note on terminology and notation: if $\Sigma' \to \Sigma$ is a regular cover and $\pi_1(\Sigma') < \pi_1(\Sigma)$ is a characteristic subgroup, we say that the cover is a **characteristic cover**. We call the quotient of a group by a characteristic subgroup a **characteristic quotient**. If G is a group we will write G^{ab} for its **abelianization**. It is standard that $G^{ab} = H_1(G, \mathbb{Z})$, and we will use these two notations interchangeably, favoring G^{ab} in group theoretic contexts and the homology notation in more topological contexts.

Our first main result is:

Theorem 1.1. *Let $\psi \in \mathrm{Mod}^1(\Sigma)$, let Γ be a finite characteristic quotient of $\pi_1(\Sigma)$, and let Σ_Γ denote the associated covering space of Σ. If $1 \neq \psi \in \mathrm{Aut}(\Gamma)$ then ψ acts nontrivially on $H_1(\Sigma_\Gamma, \mathbb{Z})$.*

We may compare Theorem 1.1 with Andersen's primary result in [A]. He considers a sequence of finite–dimensional representations of $\mathrm{Mod}(\Sigma)$, called the **quantum $SU(n)$ representations**, and he shows that these representations are asymptotically faithful as n tends to infinity. A feature of Andersen's result which we may contrast with the present one is that under Andersen's representations, Dehn twists are sent to elements of finite order, whereas we shall show that Dehn twists always act with infinite order on a sufficiently complicated abelian cover.

In order to give meaning to Theorem 1.1, we need to see that the fundamental group of a surface admits many characteristic quotients. To produce such quotients, one can take the intersection of all subgroups of a given index. Since the fundamental group of the surface is finitely generated, there are only finitely many such subgroups, so that their intersection has finite index. Another good way to produce characteristic quotients is to take towers of modulo p homology covers for some prime p. The advantage of the latter construction is that then the deck group is a p–group and therefore has simpler structure than a general finite group.

When we decorate a cover with an adjective such as solvable, nilpotent, etc. we mean that the deck group has this property. The fundamental observation is that if we take a sequence of quotients which exhaust $\pi_1(\Sigma)$, then each automorphism of $\pi_1(\Sigma)$ acts nontrivially on one of the quotients. We immediately obtain:

Corollary 1.2. *Let $\{\Sigma_i\}$ be a sequence of exhausting, finite characteristic covers of Σ and let $1 \neq \psi \in \mathrm{Mod}^1(\Sigma)$. Then ψ induces a nontrivial automorphism of $H_1(\Sigma_i, \mathbb{Z})$ for some i. In particular, we may assume ψ acts nontrivially on the homology of some solvable, nilpotent, or p–cover for any prime p.*

Let $\Sigma' \to \Sigma$ be a characteristic cover with deck group Γ. It is not immediately clear that the action of ψ on $H_1(\Sigma', \mathbb{Z})$ will not coincide with the action of some element of Γ. To this end, we have the following extension of Theorem 1.1:

Theorem 1.3. *Let $\psi \in \mathrm{Mod}^1(\Sigma)$. Suppose that for every finite characteristic cover $\Sigma' \to \Sigma$ the action of ψ on $H_1(\Sigma', \mathbb{Z})$ coincides with that of an inner automorphism. Then ψ induces an inner automorphism of $\pi_1(\Sigma)$. In particular there is a finite cover of Σ where each lift of ψ is nontrivial.*

Theorem 1.1 can be viewed as positive evidence towards a question which McMullen asked the author. McMullen has since answered the question in the strongest negative sense possible in [Mc]. To state the question properly, we first develop some notation. Let $\psi \in \mathrm{Mod}(\Sigma)$ be a pseudo-Anosov mapping class and let $K_\psi = K$ be its **geometric dilatation**. A reader unfamiliar with pseudo-Anosov homeomorphisms should consult a general reference on mapping class groups, for instance chapter 13 of [FM] or exposé 10 of [FLP]. The geometric dilatation of the pseudo-Anosov map ψ is the exponential of the entropy for the smallest entropy representative of ψ in its isotopy class. Consider the collection of ψ–invariant finite covers $\{\Sigma'\}$ of Σ. Our identification $\mathrm{Mod}^1(\Sigma) < \mathrm{Aut}(\pi_1(\Sigma))$ gives us a lift of ψ to each Σ', and we let $K_H(\psi, \Sigma')$ be the **homological dilatation** of ψ, namely its spectral radius as an automorphism of $H_1(\Sigma', \mathbb{R})$. Since K and K_H are can be defined in terms of word growth in groups, it follows that K is constant over this family and both K_H and K are independent of the choice of lift of a mapping class to each cover (cf. [FLP], exposé 10). McMullen's question can be stated as:

Question 1.4. *Is*

$$\sup_{\Sigma' \to \Sigma} K_H(\psi, \Sigma') = K?$$

The original motivation for this question was the work of Kazhdan in [Ka] and also Rhodes in [Rh], where it is shown that a hyperbolic metric on a Riemann surface can be recovered from the Jacobians of its finite covers. An analogous question in Teichmüller theory is to determine whether or not the Teichmüller metric on the moduli space of curves can be recovered from the Kobayashi metric on the moduli space of principally polarized abelian varieties. McMullen's result implies a negative answer to this question. Precisely, he proves:

Theorem 1.5. *Suppose that ψ is a pseudo-Anosov homeomorphism of a surface with dilatation K. Then either K is the spectral radius of the action of ψ on a finite cover of Σ, or there is an $0 \le \alpha < 1$ such that*

$$\sup_{\Sigma' \to \Sigma} K_H(\psi, \Sigma') = \alpha K.$$

Even though Question 1.4 is completely answered, one can still understand the action of $\mathrm{Mod}^1(\Sigma)$ on the homology of finite covers and see what information becomes available. We will see later that we can produce the bound

$$\sup_{\Sigma' \to \Sigma} K_H(\psi, \Sigma') \le K$$

by general methods which have little to do with the context of surfaces and mapping classes. We will prove:

Theorem 1.6. *Let M be a compact manifold equipped with a metric which is compatible with the manifold topology, and let ψ be a K–Lipschitz homeomorphism of M. Let $K_{H,i}$ denote the homological dilatation of ψ on the i^{th} homology with real coefficients. Then $K_{H,i} \le K^i$.*

We now wish to see how homology representations of mapping classes reveal more of their properties. Recall that the **Nielsen–Thurston classification** of mapping classes says that each mapping class $\psi \in \mathrm{Mod}(\Sigma)$ can be classified as either **finite order**, **reducible** or **pseudo-Anosov**. The final main result of this paper is that the homology of all finite covers of Σ is sufficiently rich to reveal the Nielsen–Thurston classification of each mapping class. Before stating a precise result, we need to give some more setup and background.

The homology representation of $\mathrm{Mod}(\Sigma)$ can determine whether certain mapping classes are pseudo-Anosov. Recall that there is a well–known **Casson–Bleiler criterion** which can certify that certain mapping classes are pseudo-Anosov by looking at their characteristic polynomials of their action on the homology of the surface (see page 75 of [CB]). The Casson–Bleiler criterion is silent on Torelli mapping classes, so some other ideas are necessary.

All the results of this paper, including the homological version of the Nielsen–Thurston classification, can be stated in terms of covering spaces of the base surface, but there is a way to encode the homologies of sequences of covers and their mapping class group actions in one single gadget. The Nielsen–Thurston classification of a mapping class can either be determined by sampling infinitely many covering spaces of the base surface while keeping track of some sort of "compatibility" of homology classes, or by considering just one representation. In the end, the two methods are not fundamentally different but we favor the latter for the sake of a somewhat simpler statement. We now define the relevant representation:

Let F be a field which we assume to have characteristic zero in order to avoid technical difficulties. Let Σ' be a finite characteristic cover of Σ. We suppose that Σ' is an element in a class $\mathcal{K}$ of finite characteristic covers of Σ. We may define the **pro–$\mathcal{K}$ F–homology** of Σ by taking

$$H(\Sigma) = \varprojlim H_1(\Sigma', F),$$

as Σ' ranges over all elements of $\mathcal{K}$. The field of coefficients and $\mathcal{K}$ will generally be apparent from context and we will suppress them from the notation. We write $\widehat{G}$ for the pro–$\mathcal{K}$–completion of $\pi_1(\Sigma)$. It turns out that $H(\Sigma)$ is a free module over $\widehat{G}$, but we will not need this fact. In this context, Theorem 1.1 can be restated as saying that if $\mathcal{K}$ exhausts $\pi_1(\Sigma)$ then $H(\Sigma)$ is a faithful representation of $\mathrm{Mod}^1(\Sigma)$.

Note that $H(\Sigma)$ comes equipped with a natural action of $\widehat{G}$. It is easy to check that this action is well–defined and compatible with the action of $\mathrm{Aut}(\pi_1(\Sigma))$. We can now use the representation $H(\Sigma)$ to detect the Nielsen–Thurston classification of mapping classes. To state a precise result, we need some setup which applies to general faithful representations of $\mathrm{Mod}(\Sigma)$:

Let Γ be a finite characteristic quotient of $\pi_1(\Sigma)$. If ρ is a finite-dimensional representation of Γ over an algebraically closed field of characteristic zero, we may decompose the representation as a direct sum of irreducible modules. If ρ is an irreducible representation of Γ and $\alpha \in \mathrm{Aut}(\Gamma)$, we get a new representation of Γ via $\rho \circ \alpha$. To each representation ρ we can associate its character χ. $\mathrm{Aut}(\Gamma)$ acts on the characters of Γ by precomposition. We say two characters χ and χ' are **equivalent** on a cyclic subgroup $\langle g \rangle < \Gamma$ if χ and χ' are equal as functions on $\langle g \rangle$, and inequivalent otherwise.

Theorem 1.7. *Let $H(\Sigma)$ be the pro–$\mathcal{K}$ complex homology, where $\mathcal{K}$ is the class of all characteristic finite covers of Σ. The action of $\psi \in \mathrm{Mod}(\Sigma)$ on the finite representations of $\widehat{G}$ in $H(\Sigma)$ determines the Nielsen–Thurston classification of ψ as follows:*

1. *The mapping class ψ is finite order if and only if some power ψ^n of ψ induces the trivial automorphism of the representations of the deck group for every $\mathcal{K}$–cover of Σ.*
2. *The mapping class ψ is reducible if and only if there is a power ψ^m of ψ and a nonperipheral homotopy class $1 \neq g \in \pi_1(\Sigma)$ such that on every finite $\mathcal{K}$–cover of Σ and every character χ of the deck group, χ and $\chi \circ \psi^m$ are equivalent on the image of $\langle g \rangle$ in the deck group.*
3. *The mapping class ψ is pseudo-Anosov if and only if the previous two conditions fail.*

The integers n and m above are the order of ψ and the smallest power of ψ needed to force a reducible mapping class to fix an isotopy class of curves, respectively.

Note that even though the conditions of Theorem 1.7 use the action of $\mathrm{Mod}^1(\Sigma)$ on certain quotients of $\pi_1(\Sigma)$, the characterization of the mapping classes does actually exploit the action of automorphisms on the homology of finite covers. This is because every representation of the deck group of a finite cover of Σ occurs as a summand of the homology of the cover. The homological Nielsen–Thurston classification uses the fact that each mapping class permutes the representations of the deck group on the homology of each finite cover in a way which reveals its Nielsen–Thurston classification.

We will formulate and prove an analogous result for automorphisms of free groups in Section 6.

The terminology of the Nielsen–Thurston classification is reminiscent of representation-theoretic terminology, and it is interesting to understand the relationship between topology and representation theory. The fact which Theorem 1.7 makes precise is that a mapping class is reducible if and only if the associated action of ψ on $H(\Sigma)$ has (more or less) a compatible family of fixed subrepresentations in the homologies of all finite characteristic covers of Σ. It is natural to ask whether the following strengthened version of Theorem 1.7 is true: ψ is pseudo-Anosov if and only if there is a finite cover of Σ for which ψ together with the deck transformation group act irreducibly on the rational homology. This is not true in a very strong sense:

Theorem 1.8. *Let $\psi \in \mathrm{Mod}^1(\Sigma)$ and $\Sigma' \to \Sigma$ a finite regular cover with deck group Γ which is ψ–invariant. Suppose that ψ acts irreducibly on $H_1(\Sigma', F)$, where F is a field of characteristic zero. Then $\Sigma' \to \Sigma$ is the trivial covering map. Furthermore, we may replace ψ by a nonzero power after which ψ preserves the isotypic component of every irreducible Γ–module over F occurring in $H_1(\Sigma', F)$.*

We recall that if ρ is a representation of a finite group Γ which is decomposed into irreducible Γ–modules, the **isotypic component** of a particular irreducible representation λ is the direct sum of all copies of λ occurring in ρ.

In particular, every pseudo-Anosov homeomorphism acts reducibly on every nontrivial finite cover of Σ, and any particular infinite order mapping class can be replaced by a power which acts with a prescribed number of invariant subspaces on some finite cover. Theorem 1.8 should be thought of as a dramatic failure of a straightforward generalization of the Casson–Bleiler criterion.

2. Acknowledgements

The idea for this paper came out of numerous discussions with C. McMullen. The author thanks J. Birman for her patience and help with various incarnations of this paper. The author also thanks N. Avni, B. Farb, S. Friedl, E. Hironaka, D. Margalit, and A. Putman for various helpful conversations. The author thanks A. Pettet and J. Souto for inspiring the material contained in the Section 6. Finally, the author thanks the anonymous referees for the time they spent thoroughly understanding this work, and for their numerous helpful, insightful and thorough comments and suggestions. The author is partially supported by a NSF Graduate Research Fellowship.

3. Generalities on residual finiteness and automorphisms

In this section, let G be a finitely generated residually finite group. The following is well–known and originally due to Baumslag:

Lemma 3.1. $\mathrm{Aut}(G)$ *is residually finite.*

Proof. If $H < G$ is a finite index subgroup, then G admits a finite index characteristic subgroup contained in H. This claim uses the fact that G is finitely generated. If $\alpha \in \mathrm{Aut}(G)$ is nontrivial, then there is a $g \in G$ such that $g \neq \alpha(g)$. Since G is residually finite, there is a finite index subgroup H of G which contains neither g nor $\alpha(g)g^{-1}$. There is a finite index characteristic subgroup $H' < H < G$, and α descends to an automorphism of G/H'. Since g and $\alpha(g)g^{-1}$ are both nonidentity elements of G/H', we have that $\alpha(g) \neq g$ in G/H'. It follows that α is nontrivial in a finite quotient of $\mathrm{Aut}(G)$. □

It is not true that if G is residually finite then $\mathrm{Out}(G)$ is necessarily residually finite. By the work of Wise in [W], any finitely generated group is a subgroup of $\mathrm{Out}(G)$ for some group G.

We will need the following observation to prove that for each mapping class ψ, there is a finite cover each lift of ψ is nontrivial. A similar argument can be found in the work of Grossman in [G]:

Lemma 3.2. *Let G be a finitely generated residually finite group. The following three statements are equivalent:*

1. $\mathrm{Out}(G)$ *is residually finite.*
2. *The closure of the image of G in $\widehat{\mathrm{Aut}(G)}$ is separated from all non-inner automorphisms.*
3. *Any non–inner automorphism of G descends to a non–inner automorphism of some finite quotient of G.*

Proof. Since G is residually finite, we have that $\mathrm{Aut}(G)$ injects into its profinite completion $\widehat{\mathrm{Aut}(G)}$. Furthermore, $\widehat{\mathrm{Aut}(G)}$ contains a natural copy of $G/Z(G)$, where $Z(G)$ is the center of G.

To say that α coincides with an inner automorphism on every characteristic quotient of G is simply to say that α is an accumulation point of the image of G under the composition

$$G \to \mathrm{Aut}(G) \to \widehat{\mathrm{Aut}(G)}.$$

Thus, α cannot be separated from all inner automorphisms on any finite quotient of G if and only if the closure of image of G in $\widehat{\mathrm{Aut}(G)}$ contains α.

Taking the topological quotient of $\widehat{\mathrm{Aut}(G)}$ by the closure of the image of G gives a Hausdorff space X. It is evident that X is the profinite completion of $\mathrm{Out}(G)$ with respect to a cofinal sequence of finite index subgroups. $\mathrm{Out}(G)$ is residually finite if and only if $\mathrm{Out}(G)$ injects into its profinite completion, and this happens only if the profinite topology on $\mathrm{Out}(G)$ is Hausdorff. Note that cosets of finite index subgroups of $\mathrm{Aut}(G)$ form a basis for the profinite topology on $\mathrm{Aut}(G)$. Therefore α will be separated from the identity in the profinite completion of $\mathrm{Out}(G)$ if and only if some coset of a finite index subgroup of $\mathrm{Aut}(G)$ separates α from the image of G. The claim follows. □

It is well–known that both $\mathrm{Out}(F_n)$ and $\mathrm{Mod}(\Sigma)$ are residually finite (see [G], also [FM], page 188). The fact that surface and free groups are residually finite is an easy fact which can be found on page 14 in [LySch], for instance.

Corollary 3.3. *Let ψ be an automorphism of a surface group or a free group G. Then there is a finite characteristic quotient Γ of G such that $\psi \in \mathrm{Aut}(\Gamma)$ does not coincide with any inner automorphism of Γ.*

Let G be a group which is residually $\mathcal{K}$. We say that G is **conjugacy separable** with respect to the class of $\mathcal{K}$–groups if for every pair $g, h \in G$ of representatives from different conjugacy classes, there is a $\mathcal{K}$–quotient $K_{g,h}$ of G such that the images of g and h in $K_{g,h}$ are not conjugate. We will need the following well–known result:

Lemma 3.4 (cf. [LySch], page 26)**.** *Let G be a free or a surface group. Then G is conjugacy separable with respect to the class of finite p–groups.*

4. An asymptotically faithful representation of $\mathrm{Mod}^1(\Sigma)$ and highly reducible actions on virtual homology

In this section, we shall prove Theorem 1.1. There are several proofs, and we will restrict ourselves to the simplest.

Lemma 4.1 (cf. [CW], [KS])**.** *Let G be a free group of finite rank or a finitely generated surface group, and let Γ be a finite quotient of G with kernel K. Then Γ acts faithfully on $H_1(K, \mathbb{Z})$.*

Proof. Let X, Y be a $K(G, 1)$ and a $K(K, 1)$ respectively. If $\gamma \in \Gamma$ is nontrivial, then γ acts fixed-point freely on Y, so the Lefschetz number of γ must be zero. Let $n = |\Gamma|$. Homology in degrees zero and two is easy to describe, and γ acts trivially on them. In this way, we obtain the character χ of the representation of Γ on $H_1(Y,\mathbb{Z})$: we have that

$$\chi(1) = 2n(g-1)+2$$

or

$$\chi(1) = n(d-1)+1,$$

depending on whether X is homotopy equivalent to a surface of genus g or a wedge of d circles, and $\chi(\gamma) = 2$ or 1 respectively when $\gamma \neq 1$. Since characters (and $\chi(1)$ in our case) uniquely determine the isomorphism class of a representation, it follows that the representation of Γ consists of $(2g-2)$ copies of the regular representation and two copies of the trivial representation (respectively $(d-1)$ and one). □

The proof of Lemma 4.1 is stronger than what we need for Theorem 1.1, but we will use this extra information later. Lemma 4.1 also follows from the fact that Γ is a finite group of automorphisms and hence Γ acts nontrivially on the homology of the corresponding cover (cf. [FM], page 186).

Lemma 4.1 can be seen as a generalization of the following easy result which was one of the original motivations for the proof of Theorem 1.1:

Lemma 4.2. *Let $\gamma \in G$, where G is free or a surface group as above. Then there is a finite p–group quotient of G with kernel K such that $0 \neq [\gamma] \in K^{ab}$.*

Proof. Suppose γ is homologically trivial. Then there is a finite p–group quotient P of G with center $Z(P)$ such that $1 \neq \gamma \in Z(P)$. Consider $\Sigma_{P/Z(P)}$, the cover of the base surface corresponding to $P/Z(P)$. We set K to be the kernel of the map $G \to P/Z(P)$, so that $\pi_1(\Sigma_{P/Z(P)})$ is canonically identified with K. Since $\pi_1(\Sigma_{P/Z(P)})$ admits $Z(P)$ as a quotient, it follows that γ is nontrivial in $\pi_1(\Sigma_{P/Z(P)})^{ab}$, whence the claim. □

We remark briefly that if $c \subset \Sigma$ is a simple closed curve, one can explicitly produce covers where the homology class of each lift of c is nontrivial. If c is nonseparating, then its homology class is already nontrivial. Therefore we may assume that c separates Σ, and c is determined up to a homeomorphism of Σ by the splitting on $H_1(\Sigma,\mathbb{Z})$ it determines. Suppose that Σ is closed. Clearly we may arrange Σ so that its "holes" are linearly ordered, and c lies somewhere between the first and the last hole. Take two nonseparating simple closed curves which travel "through the hole", one for first and and one for the last hole. We may assume that these curves miss c. There is a finite cover $\Sigma' \to \Sigma$ given by counting the sum of the algebraic intersection numbers with each of these curves modulo 2. It is easy to verify that c has two distinct lifts to Σ', and that each of them is nonseparating. It is trivial to generalize this construction so that the covering has degree n for any positive integer n.

We also note that a similar construction holds when Σ is not closed and has a finite set P of punctures. The modification we need to perform is to do intersection theory in $H_1(\Sigma,P,\mathbb{Z})$ and use modular algebraic intersection numbers with cycles in $H_1(\Sigma,P,\mathbb{Z})$ to find the desired covers.

Lemma 4.1 shows that if $K < \pi_1(\Sigma) = G$ is a finite index normal subgroup and ρ is an irreducible representation of $\Gamma = \pi_1(\Sigma)/K$ over a field F of characteristic zero of dimension n_ρ, then ρ occurs in the representation of Γ on $K^{ab} \otimes F$ with multiplicity $(2g-2)n_\rho$ when G is not free and ρ is nontrivial, with multiplicity $(d-1)n_\rho$ when G is free and ρ is nontrivial, and with multiplicity $\operatorname{rk} G$ when ρ is trivial. We can use this observation to prove that in general, the action of a mapping class on the homology of a finite cover is highly reducible:

Proof of Theorem 1.8. Let $\Sigma' \to \Sigma$ be the cover in question. Over F, we always have the F–homology of Σ identified as a subspace of $H_1(\Sigma', F)$, which we call the pullback. If ψ is a mapping class of Σ then ψ clearly preserves the pullback of the homology of the base on any ψ–invariant cover. Thus we see that if $\Sigma' \to \Sigma$ is nontrivial and characteristic, the action of ψ on $H_1(\Sigma', F)$ is reducible.

For the second claim of the Theorem, we have that some power of ψ induces the identity automorphism of the deck group Γ. It follows that ψ preserves all the isotypic components of Γ–modules occurring in $H_1(\Sigma', F)$. □

The method of the previous proof is soft and applies to groups in general. It seems that whenever ψ is an automorphism of a group with at least some finite index ψ-invariant subgroups, then the action of ψ on the homology of these subgroups will be highly reducible.

Using Lemma 4.1, we can now give a proof of Theorem 1.1.

Proof of Theorem 1.1. Let Γ be as in the statement of the theorem and suppose the contrary, so that $\psi(d) = d$ for all $d \in H_1(\Sigma_\Gamma, \mathbb{Z})$. Choose d which rationally generates a regular representation, which we can assume to be an integral class by replacing it with a multiple if necessary. If $\gamma \in \Gamma$, we have

$$\gamma \cdot d = \psi(\gamma \cdot d) = \psi(\gamma) \cdot d,$$

which is a contradiction since d generates a regular representation. □

An immediate corollary of the proof of Theorem 1.1 is the following general statement:

Corollary 4.3. *Let G be a residually finite group and suppose that each $1 \neq g \in G$ acts nontrivially by conjugation on the homology of some finite index subgroup G'_g of G. Then each $\psi \in \operatorname{Aut}(G)$ acts nontrivially on the homology of some finite index subgroup G'_ψ of G.*

After some conversations with the author about early drafts of this paper, S. Friedl independently found a proof of Theorem 1.1 which is essentially the same. The various corollaries to Theorem 1.1 follow from the fact that G is residually solvable, nilpotent, p, etc.

For certain mapping classes one can explicitly exhibit finite covers of Σ to which these mapping classes lift and act nontrivially on the integral homology of the cover. We can do this explicitly for any Dehn twist, and the idea is identical to the lifting of separating curves to nonseparating ones:

Proposition 4.4. *Let p be a prime and $c \subset \Sigma$ a simple closed curve on Σ. Let T_c denote a Dehn twist about c. Then there is a degree p cover Σ_p of Σ*

to which T_c lifts and acts with infinite order on $H_1(\Sigma_p, \mathbb{Z})$ when $p \neq 2$, and a degree 4 cover with this property otherwise.

Proof. We may evidently assume that c is separating. Construct a cover Σ_p to which c lifts to p nonseparating curves. It is clear that when $p > 2$ that the simultaneous lifted Dehn twist about these curves acts nontrivially on the homology of Σ_p. This can be easily seen by constructing a nonseparating curve which intersects only 2 of the lifts of c.

When $p = 2$, Σ_p will not do the trick since then T_c lifts to a twist about a bounding pair, which is still an element of the Torelli group. By taking a 4–fold cover, we find T_c lifts out of the Torelli group. □

It is conceivable that the methods of the previous proposition could be used to give another more topological proof of Theorem 1.1, but there seem to be many technical difficulties arising due to twists about single curves lifting to twists about multicurves. We can easily see the following, however:

Corollary 4.5. *Let $\psi \in \mathrm{Mod}(\Sigma)$ be a Dehn twist about a simple closed curve, or more generally a product of Dehn twists about a multicurve in Σ. Then ψ acts with infinite order on the homology of a finite cover of Σ. Fixing any prime p, we may assume that the cover is an abelian p–power cover. In particular, there exists a finite cover of Σ where each lift of ψ has infinite order.*

Recall that if ψ is nontrivial in $\mathrm{Mod}(\Sigma)$ then we can lift ψ to an automorphism of $\pi_1(\Sigma)$ and find a finite quotient Γ of $\pi_1(\Sigma)$ on which ψ acts by a non–inner automorphism. We thus obtain the extension of Theorem 1.1 which we mentioned in the introduction:

Corollary 4.6. *Each homeomorphism of Σ which is not isotopic to the identity acts nontrivially on the homology of some finite cover Σ' of Σ. Furthermore, we may choose the cover so that every lift of the homeomorphism acts nontrivially on the homology of the cover.*

Proof. There is a finite characteristic quotient Γ of $\pi_1(\Sigma)$ which has the property that $\psi(g)$ is not conjugate to g for any $g \in \Gamma$ since $\mathrm{Mod}(\Sigma)$ is residually finite. It follows that there is a $d \in H_1(\Sigma, \mathbb{Z})$ such that $\psi(g \cdot d) \neq (h^{-1}gh) \cdot d$ for any $h \in \Gamma$. □

The exact same argument holds for non–inner automorphisms of free groups since $\mathrm{Out}(F_n)$ is residually finite for each n.

5. The Nielsen-Thurston classification of mapping classes and homology

We are now in a position to give the proof of the Nielsen-Thurston classifications.

Lemma 5.1. *Let $g \in \pi_1(\Sigma)$ be a nonperipheral nonidentity element, and suppose that $\psi \in \mathrm{Mod}^1(\Sigma)$ has infinite order in $\mathrm{Mod}(\Sigma)$ and preserves the conjugacy class of g. Then ψ is reducible.*

Proof. Let γ be a geodesic representative of the free homotopy class of g. Fixing a hyperbolic metric on Σ, we consider the length of the geodesic representative of $\psi^n(\gamma)$ as a function of n. If ψ is pseudo-Anosov, we have that this function grows like K^n, where K is the dilatation of ψ (see [FLP], exposé 10 by Fathi and Shub). If ψ preserves the conjugacy class of g then this function is constant, a contradiction. □

Proof of Theorem 1.7. Note that if ψ is inner then ψ preserves all of the representations of the deck group on any cover. It follows that the action of $\mathrm{Mod}^1(\Sigma)$ on the representations of the deck group of each cover descends to an action of $\mathrm{Mod}(\Sigma)$.

If the conjugacy class of $c \in \pi_1(\Sigma)$ is invariant under ψ then we see by Lemma 5.1 that ψ is reducible. If $\psi(c)$ is conjugate to c, then for each character of the deck group Γ, we have that $\chi(\psi(c)) = \chi(c)$. Conversely, if ψ does not preserve the conjugacy class of c then this becomes visible on some cover of Σ since $\pi_1(\Sigma)$ is conjugacy separable. In particular, there will be a finite characteristic quotient Γ of $\pi_1(\Sigma)$ such that the images of c and $\psi(c)$ are not conjugate. Since the irreducible characters over $\mathbb{C}$ form a basis for the class functions of Γ and since every irreducible representation of Γ occurs as a summand of $H_1(\Sigma_\Gamma, \mathbb{C})$, we see that there is a character of χ such that $\chi(c)$ and $\chi(\psi(c))$ do not coincide. Thus, the finite characters of $\pi_1(\Sigma)$ detect whether or not ψ is reducible. The characterization of finite order mapping classes is clear. The conclusion of the theorem follows. □

6. Free groups and detecting the classification of free group automorphisms

Analogously to the Nielsen–Thurston classification, there is a classification of free group automorphisms. An (outer) automorphism ϕ of the free group on n generators F_n is called either **finite order**, **reducible** or **irreducible**. The reader who is not familiar with this classification seeking an accessible introduction to the classification should consult Bestvina's article [Be].

By analogy with the construction of $H(\Sigma)$, we may take an exhausting inverse system $\mathcal{K}$ of finite index characteristic subgroups of F_n, abelianize these groups simultaneously, tensor with the field F and take the inverse limit. Let us call the resulting vector space $H(F_n)$. We have:

Corollary 6.1. *The action of $Aut(F_n)$ on $H(F_n)$ is faithful.*

The proof of this corollary is identical to the proof of Theorem 6.2. The main result of this section is the following:

Theorem 6.2. *The representation $H(F_n)$ with complex coefficients detects irreducible automorphisms and finite order automorphisms.*

We will need to appeal to the following characterization of reducible automorphisms which can be found in [BH]:

Lemma 6.3. *Let $\phi \in Out(F_n)$. Then ϕ is reducible if and only if there are free factors F_{n_i}, $1 \leq i \leq k$, $n_1 < n$, such that $F_{n_1} * \cdots * F_{n_k}$ is a free factor of F_n and ϕ cyclically permutes the conjugacy classes of the $\{F_{n_i}\}$.*

Let ψ be a reducible automorphism of F. By Lemma 6.3, we may assume that there is a free factor decomposition $A * B$ of F such that the conjugacy class of A is preserved by ψ. If $a \in A$ is any particular element, we may lift ψ to an automorphism of F which sends a into A. If ψ is irreducible then for any candidate free decomposition of $F = A * B$, we can find an $a \in A$ such that $\psi(a)$ is not conjugate to an element of A. The next lemma shows the finite index subgroups of F_n detect the failure of a free splitting of F_n to be preserved by an automorphism.

Lemma 6.4. *Write $F_n = A * B$. Suppose $x \in F_n$ is not conjugate to any element of A. Then for every $a \in A$ there is a finite quotient of F_n such that image of x is not conjugate to a. Fixing a prime p, we may assume that this quotient is a p–group.*

Proof. This follows from the conjugacy separability of the free group (see [LySch]). The free group is conjugacy separable with respect to the class of finite p–groups, whence the second claim. □

Proof of Theorem 6.2. Let ψ be irreducible and $A * B$ any candidate splitting. Lift ψ to $\mathrm{Aut}(F_n)$. Let $F' < F_n$ be any characteristic subgroup with deck group Γ. By Lemma 4.1, we have that Γ acts on $(F')^{ab} \otimes \mathbb{C}$, and that each irreducible representation of Γ occurs as a direct summand. Fix $a \in A$ such that $\psi(a)$ is not conjugate to any element of A. For each $a' \in A$, there is a characteristic quotient Γ of F_n and an irreducible character χ of Γ such that $\chi(\psi(a))$ and $\chi(a')$ do not coincide (we are appealing strongly to the fact that we are doing representation theory over $\mathbb{C}$). It follows that the representations of the deck groups of all finite characteristic quotients of F_n on the complex homology of finite index subgroups witnesses the fact that ψ does not preserve $A * B$.

If ψ has finite order, then replacing ψ by a power and lifting to $\mathrm{Aut}(F_n)$ shows that for each finite quotient Γ of F_n, each $x \in \Gamma$, and each irreducible character χ of Γ, $\chi(\psi(x)) = \chi(x)$. By Grossman's work in [G], it follows that ψ must be inner, since any automorphism of a free group which preserves the conjugacy class of every element of the free group is inner. □

7. Using representations to approximate K_ψ

Though we have obtained a representation–theoretic characterization of the Nielsen–Thurston classification and a faithful representation of the mapping class group, the objects in this paper are difficult to work with. Understanding the finite nilpotent covers of a thrice–punctured sphere is no easier than understanding all two–generated finite nilpotent groups. Therefore, even the simplest examples present a lot of difficulty in the setup we consider here.

We can say a few things, though. For instance, consider the braid groups B_n, identified with the the mapping class groups of n–times punctured disks. There is a natural homological representation to consider here: the Burau representation. Indeed, we have a homomorphism from $F_n \to \mathbb{Z}$ that takes a word in a fixed standard generating set for F_n to its exponent sum. This homomorphism gives rise to a covering space X called the **Burau cover**, and by taking values in $\mathbb{Z}/m\mathbb{Z}$ we get the **modulo** m **Burau cover**. The modulo m Burau covers together form the **finite Burau covers**. These obviously correspond to covers where small loops about the punctures are all unwound the same number (namely m) of times. The braid group acts on F_n, commuting with this homomorphism. We thus get a representation of the braid group on the covering corresponding to the kernel of this homomorphism, which is the classical Burau representation, V_n. It is well known that V_3 is faithful and that V_n is not faithful for $n \geq 5$ (see [B2] and the references therein). The moment V_n is not faithful, there are pseudo-Anosov mapping classes whose nontriviality is not detected by the Burau representation, much less their dilatations. This follows from the fact that any non–central normal subgroup of the mapping class group contains pseudo-Anosov homeomorphisms in each of its cosets. Conversely, a representation that contains no pseudo-Anosov classes in its kernel is faithful. Since the kernel and image of the homomorphism are B_n invariant, we see that if X is the cover of Σ corresponding to the kernel of the homomorphism, B_n acts on $H_1(X, \mathbb{Q})$ by $\mathbb{Z}[t^{\pm 1}]$–linear maps. We can set the parameter to be a root of unity and thus obtain a representation of the braid group on the homology of the multiply punctured disk with twisted coefficients. Such homology groups arise naturally from covering spaces. Letting t be a primitive n^{th} root of unity gives rise to some of the twisted homology given by the modulo n Burau cover. It is fairly easy to see the following proposition:

Proposition 7.1 (cf. [BaBo]). *Let $\psi \in B_n$ be a pseudo-Anosov braid and S^1 denote the unit complex numbers. Then*

$$\sup_{t \in S^1} \rho(V_n(\psi)) \leq K,$$

and the supremum represents the supremum of homological dilatations taken over all finite cyclic covers of $\Sigma = \Sigma_{0,n,1}$ which have equal branching over each of the punctures.

The proof of the proposition, which we will not give here, follows roughly from Theorem 1.6 and the fact that a pseudo-Anosov map with dilatation K is K–Lipschitz with respect to the flat metric defined by the two invariant foliations.

Proof of Theorem 1.6. The proof proceeds by producing a bound for simplicial chains, and estimating the growth of the action of the homeomorphism f^n on the vector space of real simplicial chains. We can then estimate the spectral radius ρ of the action of f on the chains and then deduce an estimate of the spectral radius of the induced action on the real homology.

Choose a simplicial decomposition of X with a very fine simplices, so that any point in X is $\ll 1/K$ from the barycenter of a simplex. Then, if γ is a loop in X, we will be able to homotope it to a path lying in the 1-skeleton of X without increasing the length by more than some constant factor C that will work for all of X. Choose the subdivision also so that all the 1–simplices have approximately the same size. This can be done as follows: consider the length spectrum of all 1–simplices for some decomposition. This is a finite set since X is compact. Consider any two positive real numbers s, t. For any $\epsilon > 0$, there exist integers m, n such that $|s/m - t/n| < \epsilon$. By an easy induction, for any ϵ and any finite collection of positive real numbers, we can find a finite collection of integers such that the corresponding quotients differ pairwise by no more than ϵ.

Therefore, given any ϵ, we can subdivide the 1-skeleton of our simplicial complex so that the lengths of any two 1-simplices differ by no more than ϵ. Now let z be a 1–cycle. Writing z as a weighted union of 1–simplices, we may talk about the length ℓ of z. Let f_* denote the linear map on real simplicial chains induced by f, and consider $f_*^n(z)$ and $f^n(z)$. Since f is K–Lipschitz, the length of $f^n(z)$ is no more than $K^n \cdot \ell$. On the other hand we can homotope $f^n(z)$ to the 1–skeleton, thus increasing its length to no more than $C \cdot K^n \cdot \ell$. Choosing ϵ small enough, we see that if z required m different 1–simplices to be expressed as a 1-chain, $f_*^n(z)$ requires no more than $C/(1-\epsilon) \cdot K^n \cdot m$ simplices.

Let G be a finitely generated group, and $g : G \to H$ a surjective homomorphism. There is a well-defined length function on G given by the graph metric on a fixed Cayley graph for G, and it induces a length function on H. Furthermore, it is clear that the induced length function is bounded by the length function on ℓ_G, i.e. $\ell_G(\gamma) \geq \ell_H(g(\gamma))$ for all $\gamma \in G$. Since homology is a quotient of a subgroup of the 1–chains, we obtain $\rho_1(f_*) \leq K$.

The proof for higher homology groups is analogous. The metric gives us a way to measure the volume of simplices: the volume of a very small m–cube is given by the m^{th} power of the side lengths. Therefore, f will scale the volume element by no more than K^m. As before, we can cut up m–simplices so that their volumes are all similar. So, if z is an m-cycle, we can homotope $f^n(z)$ to sit in the m–skeleton of X, increasing its volume by no more than a factor of some constant C. The constant C can be estimated as follows: if an m–chain c intersects the interior of an m'-simplex S with $m' > m$, then we perform a homotopy $rel\,\partial S$ to push c to the $(m'-1)$–skeleton $S \cap X_{m'-1}$ and proceed inductively. Subdividing the interior of S if necessary, we can assume the homotopy does not change the volume of $c \cap S$ by much. The subdivision of the m-skeleton into simplices of similar size can be done afterwards without altering the validity of the proof. This proves the claim. □

In order to have any hope of seeing the dilatation of a pseudo-Anosov homeomorphism on a finite cover, one must be able to resolve all the odd–order singularities of an associated quadratic differential on a finite cover. Note that there is a two–sheeted covering of the thrice punctured disk which gives a thrice punctured torus with one boundary component. Any simple pole of any quadratic differential is resolved on such a cover since each pole is located at a puncture, so that we get a quadratic differential on a once–punctured torus with no poles. Resolving odd–order zeros at non–punctures is not possible without branching. To see more precisely that poles of quadratic differentials can generally be resolved to be regular points on a finite cover we have the following proposition:

Proposition 7.2. *Suppose that ψ stabilizes a Teichmüller geodesic determined by a quadratic differential $q = q(z)\,dz^2$ which has only simple poles and even order zeros. Then there is a finite unbranched cover Σ' of Σ such that the lift of q has only even order zeros. In particular, there exists a finite unbranched cover $\Sigma' \to \Sigma$ such that for any lift of ψ to Σ', the homological and geometric dilatations of ψ coincide.*

Proof. Let P be the set of punctures of Σ. At each point in P we may assume the stable foliation $\mathcal{F}$ determined by q has either an even-pronged singularity or a one-pronged singularity. In the latter case, we may simply fill in the missing point since the quadratic differential associated to $\mathcal{F}$ has a zero at that point (so that the puncture is a removable singularity). Therefore we may assume that all the points in P are poles of the associated quadratic differential. By passing to a characteristic cover of Σ, we may assume $|P|$ is even. Such a cover might be given by taking the cover associated to $H_1(\Sigma, \mathbb{Z}/2\mathbb{Z})$ if $|P| = 1$. If $|P| > 1$, label the punctures $p_1, \ldots, p_n$. Take a small loop about each puncture and record its homology class. Send the homology classes of the loops about $p_1, \ldots, p_{n-1}$ to $1 \in \mathbb{Z}/2\mathbb{Z}$. Since the sum of the the homology classes of these loops must be zero, if n is even we may send the homology class of the loop about p_n to $1 \in \mathbb{Z}/2\mathbb{Z}$. Otherwise, we are forced to send it to $0 \in \mathbb{Z}/2\mathbb{Z}$. Since the first cover is characteristic and since punctures of Σ are ψ-invariant, ψ lifts to the covers determined by these homomorphisms. After taking this cover, it is obvious that all but possibly one puncture in P will lift to a regular point of the foliation. Since the cover has even degree, the one puncture over which the cover did not branch lifts to an even number of punctures. Repeating the second cover construction, we construct a further unbranched cover (which can clearly be refined to a ψ-invariant or even characteristic cover, since these punctures are poles of the quadratic differential associated to $\mathcal{F}$) where a small loop about each puncture is unwound to half a loop about one of the punctures in the cover.

Once we have unwound each puncture as above, each pole of the quadratic differential becomes a regular point and hence can be filled in as a removable singularity. It follows that after passing to another finite cover if necessary, q becomes the global square of a holomorphic 1–form ω, whence the claim. □

We may view $B_3/Z(B_3)$ (where the center is generated by a twist about the boundary of the disk) as a subgroup of the mapping class group of the once–punctured torus by identifying it with $PSL_2(\mathbb{Z})$. The homological representation theory of this group is well–understood, especially in connection to the Nielsen–Thurston classification. The situation is already much more complicated for the four–times punctured disk. Let $\{\sigma_1, \ldots, \sigma_{n-1}\}$ denote the standard generators of the braid group B_n. Then $\beta = \sigma_1\sigma_2\sigma_3^{-1} \in B_4$ is pseudo-Anosov (see [HK] for a large class of examples in this same flavor). According to Hironaka and Kin the dilatation of β is the largest root of the polynomial

$$1 - t - 2t^2 - 2t^3 - t^4 + t^5,$$

which is approximately 2.29663. Applying the machinery of Proposition 7.1, we get that the Burau matrix $V_4(\beta)$ is

$$M = \begin{pmatrix} -t & -t^2 & -t^2 \\ 1 & 0 & 0 \\ 0 & 1 & 1 - 1/t \end{pmatrix}.$$

The characteristic polynomial of M is

$$t + u - tu + t^2u - u^2 + u^2/t + tu^2 + u^3.$$

The supremum of the spectral radii of $M = M(t)$ as t varies over S^1 is approximately 2.17401. By choosing a small mesh size, the estimate will be close to the actual value. It is possible to show that the inequality between the supremum of the spectral radii of $M(t)$ and K is strict in this case by showing that 2.17401 is already very close to the actual Burau supremum. It is a theorem of Band and Boyland in [BaBo] that the spectral radius of a Burau matrix specialized at a root of unity is either equal to the dilatation when $t = -1$ or is strictly smaller than the dilatation. Our computation shows that the difference between the spectral radii and the dilatation can be bounded away from zero. The fact that the dilatation is not achieved as the covers are allowed to vary over all covers of Σ is a consequence of [Mc].

The strict inequality of Proposition 7.1 does not change if we pass to the Lawrence–Krammer representation, which is a well–known faithful representation of the braid group. For more details and a proof that this is indeed a faithful representation of the braid group, see [B1]. Recall that the configuration space of pairs of points in a space X is the set

$$(X \times X \setminus \Delta)/(\mathbb{Z}/2\mathbb{Z})),$$

where the group action permutes the coordinates. In the case of a disk punctured at $p_1, \ldots, p_n$, the configuration space C is a 4-manifold and inherits a natural action of the braid group. The representation itself is the action of B_n on the second (usual) homology of a certain $\mathbb{Z}^2$-cover of C, viewed as a module over $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$.

We now give a verbatim account of the setup in which Bigelow works in [B1]. If α is a path in C, we may view α as $\{\alpha_1, \alpha_2\}$, where we mean unordered pairs of points. Let

$$a = \frac{1}{2\pi i} \sum_{j=1}^{n} \left(\int_{\alpha_1} \frac{dz}{z - p_j} + \int_{\alpha_2} \frac{dz}{z - p_j} \right)$$

and

$$b = \frac{1}{\pi i} \int_{\alpha_1 - \alpha_2} \frac{dz}{z}.$$

These quantities are B_n-invariant, so that B_n acts on $H_2(Z, \mathbb{Z})$ as a module over $\mathbb{Z}[t^{\pm 1}, q^{\pm 1}]$, where Z is the covering space corresponding to the map $\alpha \mapsto q^a t^b$.

Bigelow provides explicit matrices for the corresponding representation, which allows for relatively simple computation of the supremum of the homological dilatations over finite intermediate covers between C and Z (the proof of this is again analogous to Proposition 7.1). For the Hironaka–Kin example, we obtain a 6×6 matrix which we do not reproduce here. Once we obtain the supremum, we must take the square root, since the action is on second homology. Theorem 1.6 applies for the Lawrence–Krammer representation, since the action of a homeomorphism on the covering space is inherited from the action on the coordinates. If the action comes from a pseudo-Anosov homeomorphism with dilatation K, those actions can be taken to be K–Lipschitz. Theorem 1.6 guarantees that this estimate will not exceed the dilatation. We obtain the supremum $2.17433 < 2.29663$.

8. Questions and examples

A basic question which the preceding discussion leaves open is the following:

Question 8.1. *Let ψ be an infinite order mapping class. Does ψ act with infinite order on the (co)homology of some finite cover? If ψ is pseudo-Anosov, can ψ be made to act with spectral radius greater than one?*

We can formulate analogous questions to this one and Question 1.4 for outer automorphisms of free groups. For free group automorphisms, a result analogous to McMullen's Theorem is almost certainly true.

The first part of this question is really only open for automorphisms of surfaces which are generalized pseudo-Anosov, in the following sense. Given $\psi \in \mathrm{Mod}(\Sigma)$, find a canonical reduction system $\mathcal{C}$ for ψ. Then on every component of $\Sigma \setminus \mathcal{C}$, ψ either acts trivially (up to isotopy) or as a pseudo-Anosov homeomorphism, possibly after passing to a power of ψ. If there is a $\mathcal{C}' \subset \mathcal{C}$ such that ψ does a combination of Dehn twists about $\mathcal{C}'$, then we have already shown that ψ will act with infinite order on the homology of a finite cover. By a Dehn twist about $c \in \mathcal{C}'$, we mean that in $\Sigma \setminus \mathcal{C}$, c forms two boundary components in $\Sigma \setminus \mathcal{C}$, and we require that ψ restricts to the trivial map in the interiors of the corresponding components of $\Sigma \setminus \mathcal{C}$. Here we are using that $\mathcal{C}$ is indeed a canonical reduction system. Note that ψ has to have infinite order in $\mathrm{Mod}(\Sigma)$, as no inner automorphism can act with infinite order on the homology of a finite cover.

If the answer to the above question or the analogous one for free groups is true, then ψ will act with infinite order on $V_\chi^{2g-2} \subset H^1(\Sigma_\Gamma, \mathbb{Q})$ (resp. on the homology of some finite index subgroup of F_n with deck group Γ), where V_χ is an irreducible representation of Γ and V_χ^{2g-2} is its isotypic component. A rather striking illustration of this fact is the following observation:

Recall that the Torelli group $\mathcal{I}(\Sigma)$ admits a so–called Johnson filtration, written $\{J_k\}$. It can be easily shown that any element of $J_1 = \mathcal{I}(\Sigma)$ acts trivially on $\gamma_k(\pi_1(\Sigma))/\gamma_{k+1}(\pi_1(\Sigma))$ for all k (cf. [BL]). Let $\psi \in J_k \setminus J_{k+1}$. It follows that there is an element $g \in \pi_1(\Sigma)$ and $z \in \gamma_k(\pi_1(\Sigma)) \setminus \gamma_{k+1}(\pi_1(\Sigma))$ such that $\psi(g) = gz$ in $\pi_1(\Sigma)$.

Now let Σ' be a finite cover of Σ where z lifts to a nontrivial homology class and let n be such that g^n lifts to a homology class. Consider $d = g^n$ as a homology class in Σ', and suppose that $\psi(g) = g \cdot z$. It follows that $\psi(d)$ is equal to $d + z'$, where z' is the sum of the conjugates of z by all powers of g, where g is viewed as a deck transformation of the covering. The representation of the subgroup $\langle g \rangle$ where the homology class of z is located can be assumed to be a trivial representation if and only if there is a finite quotient Γ' of $\pi_1(\Sigma)$ such that both g and z are in the kernel of the quotient map, and if both g and z are nontrivial in the abelianization of the kernel.

If the conjugation action is nontrivial, then the sum of the conjugates of z is zero. Indeed, tensoring with $\mathbb{C}$ shows that g^k acts by ζ_n^k, where ζ_n is an n^{th} root of unity. Therefore, it is possible that ψ acts with infinite order on the homology of some finite nilpotent cover, but it is not obvious how to see this fact.

Finally, one would like to produce a more useful homological version of the Nielsen–Thurston classification. In the introduction, we mentioned the Casson–Bleiler criterion for a map to be pseudo-Anosov. We have seen that a naïve generalization of the Casson–Bleiler criterion cannot hold by Theorem 1.8. Malestein has used the lower central series of the surface group to detect new pseudo-Anosov homeomorphisms by constructing a generalization of the Casson–Bleiler criterion in [Ma]. We may ask the following:

Question 8.2. *Is there a characterization of pseudo-Anosov maps which is homologically visible on some finite cover? Some p–power cover?*

9. Embedding simple closed curves in $H(\Sigma)$

In order to better understand the Nielsen–Thurston classification of a mapping class ψ through its action on $H(\Sigma)$, it would be nice to encode free homotopy classes of curves inside of $H(\Sigma)$ in some natural way. In this final section, we show how this can be done with the caveat that it is not particularly useful. Precisely, we will obtain a set–theoretic injection

$$\iota : \pi_1(\Sigma) \to H(\Sigma)$$

whose image is called the set of **finite type vectors**. Recall that the fundamental group of Σ acts on $H(\Sigma)$ by conjugation. The $\pi_1(\Sigma)$–orbits of finite type vectors are in bijective correspondence with free homotopy classes of closed curves on Σ.

We will immediately obtain the following result:

Theorem 9.1. *Let $\psi \in \mathrm{Mod}(\Sigma)$ be a mapping class and let $H(\Sigma)$ be the pro–$\mathcal{K}$ F–homology with respect to the system of finite solvable covers of Σ.*

1. *A mapping class ψ has finite order if and only if some power of ψ preserves the $\pi_(\Sigma)$–orbit of each vector in $H(\Sigma)$.*
2. *A mapping class ψ is reducible if and only if some power of ψ fixes a finite type vector in $H(\Sigma)$ which corresponds to a non–peripheral element curve on Σ.*
3. *A mapping class ψ is pseudo-Anosov if and only if the previous two conditions fail.*

We now define ι precisely. Let $D_n(\Sigma)$ be the n^{th} term of the derived series of $\pi_1(\Sigma)$, so that $D_1(\Sigma) = \pi_1(\Sigma)$ and

$$D_n(\Sigma) = [D_{n-1}(\Sigma), D_{n-1}(\Sigma)].$$

For each i, choose a set of coset representatives T_i for $D_{i+1}(\Sigma)$ inside of $D_i(\Sigma)$.

Let $c \subset \Sigma$ be an essential closed curve. Fix a basepoint so that c represents an element $g \in \pi_1(\Sigma)$. Let $[g]$ be the homology class of g. We may write g uniquely as a product $t \cdot d$, where $t \in T_1$ and $d \in D_2(\Sigma)$. Now let A be any finite abelian quotient of $\pi_1(\Sigma)$, whose corresponding cover is Σ_A. We may now record the homology class of $d = t^{-1}g$ in $H_1(\Sigma_A, \mathbb{Z})$, which we write as $[d]_A$. Again, we may write $d = t'd'$, where $t' \in T_2$ and $d' \in D_3(\Sigma)$. If Σ_M is now a finite metabelian cover of Σ, we may write $[d']_M$ for the homology class represented by d'.

In general we may assign an infinite tuple

$$(t_1, \ldots, t_i, \ldots)$$

to g, where for each i we have $t_i \in T_i$. Since $\pi_1(\Sigma)$ is residually solvable, this tuple determines g uniquely. Let Γ be a k–step solvable finite quotient of $\pi_1(\Sigma)$, and let Σ_Γ be the corresponding cover. To g we assigned the tuple $(t_1, \ldots, t_i, \ldots)$, and in $H_1(\Sigma_\Gamma, \mathbb{Z})$ we consider the homology class of t_{k+1}. We write this homology class as $[g]_\Gamma$.

If Γ' is a quotient of Γ then $H_1(\Sigma_{\Gamma'}, \mathbb{Q})$ is naturally identified with a subspace of $H_1(\Sigma_\Gamma, \mathbb{Q})$. When $\mathcal{K}$ is the class of finite solvable quotients of $\pi_1(\Sigma)$, we think of $H(\Sigma)$ as the pro–solvable rational homology of Σ. By definition, $H(\Sigma)$ consists of a certain subset of the direct product of all the rational homologies of finite covers of Σ, so we can think of pro–solvable homology classes as certain infinite tuples of rational homology classes, where the entries of the tuples are in bijective correspondence with finite solvable quotients of $\pi_1(\Sigma)$. Write

$$[g]^\Gamma = \sum_{\Gamma \to \Gamma'} [g]_{\Gamma'},$$

where the sum ranges over all solvable quotients of Γ. Finally, we can define the map ι. In the entry corresponding to Γ, we define $\iota(g)$ to have the entry $[g]^\Gamma$.

We need to check that ι is well–defined. Indeed, let Γ' be a quotient of Γ. Then under the associated covering map $\Sigma_\Gamma \to \Sigma_{\Gamma'}$, it is clear that the homology class $[g]^\Gamma$ gets mapped to the homology class $[g]^{\Gamma'}$.

Let $\{H_i\}$ be the set of all finite index normal subgroups of $\pi_1(\Sigma)$ whose associated quotient groups are all at most i–step solvable. Since

$$\bigcap H_i = D_{i+1}(\Sigma),$$

we have that ι is injective.

It is clear from the definition of ι that this embedding of $\pi_1(\Sigma)$ inside of $H(\Sigma)$ is in no way algebraic and is therefore quite badly behaved from the perspective of representation theory.

The representation $H(\Sigma)$ has some rather pathological properties which make the characterization of mapping classes by their action on $H(\Sigma)$ rather subtle. For instance, it is easy to see that any mapping class which fixes a nonzero vector in $H_1(\Sigma, \mathbb{Q})$ will fix nonzero vectors in $H(\Sigma)$.

References

[A] Jørgen Ellegaard Andersen. Asymptotic faithfulness of the quantum SU(n) representations of the mapping class groups. *Ann. of Math.* 2, 163, no. 1, 347–368, 2006.

[BaBo] Gavin Band and Philip Boyland. The Burau estimate for the entropy of a braid. Available at arXiv:math/0612716, 2006.

[BL] Hyman Bass and Alexander Lubotzky. Linear–central filtrations on groups. *The mathematical legacy of Wilhelm Magnus: groups, geometry and special functions.* Contemp. Math. 169, pp. 45–98, 1994.

[Be] Mladen Bestvina. A Bers–like proof of the existence of trans tracks for free group automorphisms. arXiv:1001.0325v1 [math.GR], 2010.

[BH] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups. *Ann. of Math.*, 135 (1992), pp. 1-51.

[B1] Stephen J. Bigelow. Braid groups are linear. *J. Amer. Math. Soc.*, 14(2):471-486 (electronic), 2001.

[B2] Stephen J. Bigelow. The Burau representation is not faithful for $n = 5$. *Geom. Topol.* 3, 397–404, 1999.

[BiBu] Stephen J. Bigelow and Ryan D. Budney. The mapping class group of a genus two surface is linear. *Algebr. Geom. Topol.* 1, 699-708, 2001.

[CB] Andrew J. Casson and Steven A. Bleiler. *Automorphisms of surfaces after Nielsen and Thurston.* London Mathematical Society Student Texts, 9, 1988.

[CW] Claude Chevalley and André Weil. Über das Verhalten der Integrale 1. Gattung bei Automorphismen des Funktionenkörpers. *Abh. Math. Sem. Univ. Hamburg* 10 (1934), pp. 358–361.

[FLP] A. Fathi and F. Laudenbach and V. Poénaru. *Travaux de Thurston sur les Surfaces.* Soc. Math. de France, Paris, Astérisque 66-67, 1979.

[F] Benson Farb, editor. *Problems on mapping class groups and related topics. Proc. Symp. Pure and Applied math.*, 74, 2006.

[FM] Benson Farb and Dan Margalit. *A Primer on the mapping class group.* To be published by *Princeton University Press*, 2011.

[FP] Edward Formanek and Claudio Procesi. The automorphism group of a free group is not linear. *J. Algebra* 149, no. 2, 494–499, 1992.

[G] E.K. Grossman. On the residual finiteness of certain mapping class groups. *J. London Math. Soc.*, 9, 160–164, 1974.

[HK] Eriko Hironaka and Eiko Kin. A family of pseudo-Anosov braids with small dilatation. *Algebr. Geom. Topol.*, 6, 699–738, 2006.

[Ka] D.A. Kazhdan. On arithmetic varieties. *Lie groups and their representations* (Proc. Summer School, Bolyai Janos Math. Soc., Budapest, 1971), 151–217, Halsted, 1975.

[KS] Thomas Koberda and Aaron Michael Silberstein. Representations of Galois groups on the homology of surfaces. arXiv:0905.3002.

[LySch] Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory.* Springer, New York, 1977, reprint in 2001.

[Ma] Justin Malestein. Pseudo-Anosov homeomorphisms and the lower central series of a surface group. *Algebr. Geom. Topol.* 7, 1921–1948, 2007.

[Mc] Curtis T. McMullen. Entropy on Riemann surfaces and the Jacobians of finite covers. Preprint, 2010.

[Rh] J.A. Rhodes. Sequences of metrics on compact Riemann surfaces. *Duke Math. J.* 72, 725–738, 1993.

[W] Daniel T. Wise. A residually finite version of Rips' construction. *Bull. London Math. Soc.* 35, 1, 23–29, 2003.

Department of Mathematics, Harvard University, 1 Oxford St., Cambridge, MA 02138

E-mail address: koberda@math.harvard.edu