Exercise XII - mandatory

Math 320a/520a - Fall Semester 2017

Due Friday, 12/08/2017, 5:00 PM

1. Find a measure space $(X, \mathcal{A}, \mu)$, a subspace of $L^1(\mu)$, and a bounded linear functional f on Y with $\|f\| = 1$ such that f has two distinct extensions to $L^1(\mu)$ with unit norms.

2. Show that for every $p \in [1, \infty)$, the space $L^p([0,1])$ is separable (i.e., it has a countable dense subset), but the space $L^\infty([0,1])$ is not separable.

3. Define (for any $k \in \mathbb{N}$) $\|f\|_{C^k} = \|f\|_\infty + \|f'\|_\infty + \|f''\|_\infty + \ldots + \|f^{(k)}\|_\infty$ for each k times continuously differentiable $f : [0,1] \to \mathbb{R}$ (where $f^{(k)}$ is the k-th derivative of f). Let $C^k([0,1])$ be the collection of such functions with $\|f\|_{C^k} < \infty$. Is $C^k([0,1])$ a Banach space? Explain/prove your answer.

4. Let X be a Banach space. Prove that:

   (a) If $A \subseteq X$ is convex (i.e., $\lambda x + (1-\lambda)x \in A$ for every $x, y \in A$ & $\lambda \in \mathbb{R}$), then its closure is also convex.

   (b) The open unit ball $\{x \in X : \|x\| < 1\}$ is convex.

5. Let $C([0,1])$ be the collection of continuous complex-valued functions on $[0,1]$. Is this collection a Hilbert space with respect to $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}dx$, which is used as the inner product on L^2 (where $\bar{\cdot}$ denotes complex conjugation)?

6. Show that any Borel set $A \in \mathcal{B}_{[0,2\pi]}$ satisfies $\lim_{n\to\infty} \int_A e^{inx}dx = 0$

7. Show that if $g \in L^q(\mu)$, for a σ-finite measure μ and $1 \leq q < \infty$, then for every $\varepsilon > 0$ there exists some $\delta > 0$ such that $\|\chi_A|g|\,\|_q < \varepsilon$ whenever A is a measurable set with $\mu(A) < \delta$.

8. Let L be a linear functional on a normed linear space. Prove that L is bounded if and only if its kernel (i.e., $L^{-1}(\{0\}) = \{x \in X : L(x) = 0\}$) is closed.