Exercise XI - mandatory

Math 320a/520a - Fall Semester 2017

Due Thursday, 11/30/2017, 2:30 PM

1. Let $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ be a finite measure space with $0 < \mu(\mathbb{R}) \leq 2\pi$ such that $f(x) = \int f(x+t)d\mu(t)$ a.e. for every bounded integrable $f : \mathbb{R} \to \mathbb{R}$. Compute the value of $\mu(\{0\})$.
2. Show that L^∞, as a metric space, is complete.
3. Prove that the collection of simple functions is dense in L^p for $1 \leq p \leq \infty$.
4. Prove that the collection of continuous functions on $[-\pi, \pi]$ is dense in the space $L^2([-\pi, \pi])$ with respect to the L^2 norm on that space.
5. Show that $\|f\|_p = \sup\{\int fsd\mu : s \text{ is simple, and } \|s\|_q \leq 1\}$, where q is the conjugate exponent of $1 < p < \infty$.
6. Let $(X, \mathcal{A}, \mu)$ be a probability measure space (i.e., $\mu(X) = 1$). If $f, g : X \to [0, \infty)$ satisfy $fg \geq 1$ a.e., which of the following is satisfied by their integrals:
   (a) $(\int fd\mu)(\int gd\mu) \in (1, \infty)$
   (b) $(\int fd\mu)(\int gd\mu) \in [0, 1]$
   (c) $(\int fd\mu)(\int gd\mu) \in [1, \infty]$
7. Show that if $A, B \in \mathcal{B}_{\mathbb{R}}$ both have finite nonzero Lebesgue measures, then $\chi_A * \chi_B$ is a continuous nonnegative function, which is not identically 0.
8. Let $H : L^p \to \mathbb{C}$, $1 < p < \infty$, be a bounded linear functional (with complex values[1]) on L^p. Prove that there exists a complex-valued measurable function $g \in L^q$ (where $p^{-1} + q^{-1} = 1$) s.t. $H(f) = \int fgd\mu$ for every $f \in L^p$, and $\|H\| = \|g\|_q$.

[1] The definition of such a functional is the same as it was for the real-valued case, where here the absolute value is treated as a complex modulus.