Exercise X - mandatory

Math 320a/520a - Fall Semester 2017

Due Thursday, 11/16/2017, 2:30 PM

1. For any given $0 < \alpha < 1$, find a Borel set $E_\alpha \subseteq [-1,1]$ such that $\lim_{r \to 0^+} \frac{m(E_\alpha \cap [-r,r])}{2r} = \alpha$.

2. Show that if $(X, \mathcal{A}, \mu)$ is a finite measure space then $\|f\|_\infty = \lim_{p\to\infty} \|f\|_p$ for any measurable function $f : X \to \mathbb{R}$. Does this still hold for **integrable** functions when μ is σ-finite rather than finite? (Prove or show a counter example)

3. Prove or disprove the following statements for $1 < p < q < \infty$ and an arbitrary σ-finite measure space:
   (a) $L^p \subseteq L^q$
   (b) $L^q \subseteq L^p$

4. Give an example of an increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f' = 0$ a.e., but f is not constant on any open interval.

5. When does equality hold in:
   (a) Hölder's inequality?
   (b) Minkowski's inequality?

6. Show that if $f, g : \mathbb{R} \to \mathbb{R}$ are continuous with compact support then $f * g$ is also continuous with compact support.

7. Show that if $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, for $1 \le p < \infty$, then $\|f * g\|_p \le \|f\|_1 \|g\|_p$.

8. Consider $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ for $1 < p, q < \infty$ with $p^{-1} + q^{-1} = 1$. Prove or disprove the following statements:
   (a) $f * g$ is uniformly continuous
   (b) $\lim_{x\to\infty} f * g(x) = 0$
   (c) $\lim_{x\to-\infty} f * g(x) = 0$