Exercise IX - mandatory

Math 320a/520a - Fall Semester 2017

Due Thursday, 11/09/2017, 2:30 PM

1. State and prove the following two theorems for **signed** measures:
   (a) The Radon-Nikodym Theorem
   (b) The Lebesgue Decomposition Theorem
2. Let μ be a signed measure on $(X, \mathcal{A})$ and let $|\mu|$ be its total variation measure. Prove that there exists a measurable (w.r.t. $\mathcal{A}$) function $f : X \to \mathbb{R}$ with $|f| = 1$ μ-a.e., such that $d\mu = f d|\mu|$.
3. Let $\mathcal{A}_1 \subsetneq \mathcal{A}_2$ be two σ-algebras on X. Let μ_2 and ν_2 be two finite positive measures on $(X, \mathcal{A}_2)$ such that $\nu_2 \ll \mu_2$, and let μ_1 and ν_1 be their restrictions (correspondingly) to $(X, \mathcal{A}_1)$. Prove or disprove (with a counterexample): $\frac{d\nu_1}{d\mu_1} = \frac{d\nu_2}{d\mu_2}$ a.e. with respect to μ_1 and/or μ_2.
4. Show that if μ, ν, and λ are finite signed measures such that $\lambda \ll \nu$ and $\nu \ll \mu$, then $\lambda \ll \mu$ and $\frac{d\lambda}{d\mu} = \frac{d\lambda}{d\nu} \cdot \frac{d\nu}{d\mu}$ a.e. (with respect to μ).
5. Let $\lambda_1, \lambda_2, \ldots$ be a sequence of positive measures on $(X, \mathcal{A})$ with $\sup_n \lambda_n(X) < \infty$. Let $d\lambda_n = f_n d\mu + d\nu_n$ be the Lebesgue decomposition of λ_n (w.r.t. μ), where $\mu \perp \nu_n$ are finite positive measures, and f_n is the Radon-Nikodym derivative of the absolutely continuous (w.r.t. μ) component of the decomposition. Show that if $\lambda = \sum_{n=1}^{\infty} \lambda_n$ is a finite measure, then its Lebesgue decomposition (w.r.t. μ) is given by $d\lambda = (\sum_{n=1}^{\infty} f_n) d\mu + \sum_{n=1}^{\infty} d\nu_n$.
6. Show that if f and g are absolutely continuous on $[a, b]$ (for some $a < b$) then fg is also absolutely continuous, and furthermore, the following integration by parts formula is satisfied: $f(b)g(b) - f(a)g(a) = \int_a^b f(x)g'(x)dx + \int_a^b f'(x)g(x)dx$.
7. Show that if $F(x) = \int_a^x f(x)dx$ for some integrable $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$, then F is absolutely continuous and of bounded variation.
8. Is a function that is differentiable at each point in $[0, 1]$ necessarily absolutely continuous on this interval? Prove or give a counterexample.