Exercise VIII - mandatory

Math 320a/520a - Fall Semester 2017

Due Thursday, 11/02/2017, 2:30 PM

1. State and prove the Jordan Decomposition Theorem.

2. Consider a signed measure μ and a measurable set A. Show that A is a null set with respect to μ if and only if $|\mu|(A) = 0$.

3. Integration with respect to a signed measure μ is defined as $\int f d\mu = \int f d\mu^+ - \int f d\mu^-$. Show that $|\int f d\mu| \leq \int |f| d|\mu|$.

4. Show that if μ and ν are finite signed measures[1] on $(X, \mathcal{A})$ then $|\mu(A) + \nu(A)| \leq |\mu|(A) + |\nu|(A)$ for all $A \in \mathcal{A}$.

5. Show that if μ and ν are positive measures such that $\lambda = \mu - \nu$ is a signed measure on $(X, \mathcal{A})$, then $\mu(A) \geq \lambda^+(A)$ and $\nu(A) \geq \lambda^-(A)$ for every $A \in \mathcal{A}$.

6. Let μ and ν be two positive measures on $(X, \mathcal{A})$ such that $\nu(X) < \infty$. Show that $\nu \ll \mu$ if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\nu(A) < \varepsilon$ whenever $\mu(A) < \delta$ for all $A \in \mathcal{A}$.

7. Let μ and ν be finite positive measures on $(X, \mathcal{A})$. Show that if these measures are not mutually singular then there exists $G \in \mathcal{A}$ such that $\mu(G) > 0$ and G is a positive set with respect to the signed measure $\nu - \varepsilon\mu$ for some $\varepsilon > 0$.

8. Two measures μ and ν are called *equivalent measures* if both $\mu \ll \nu$ and $\nu \ll \mu$. Show that when μ and ν are finite, they are equivalent if and only if there exists a μ-integrable strictly positive μ-a.e. function $f = d\nu/d\mu$.

[1] A signed measure μ is finite (or σ-finite) when $|\mu|$ is finite (or σ-finite).