Exercise VII - mandatory

Math 320a/520a - Fall Semester 2017

Due Thursday, 10/26/2017, 2:30 PM

1. Let $f : [0,1]^2 \to \mathbb{R}$ be an integrable function with respect to the two-dimensional Lebesgue measure on $[0,1]^2$. Show that if $\int_0^a \int_0^b f(x,y)dydx = 0$ for all $a, b \in [0,1]$ then $f = 0$ a.e.

2. Consider two measurable spaces $(X, \mathcal{A}_X)$ and $(Y, \mathcal{A}_Y)$. Let $\mathcal{A}_X \otimes \mathcal{A}_Y$ be the product σ-algebra on $X \times Y$. Show that for every set $E \in \mathcal{A}_X \otimes \mathcal{A}_Y$ and jointly measurable (w.r.t. $\mathcal{A}_X$ and $\mathcal{A}_Y$) function $f : X \times Y \to \mathbb{R}$:

   (a) $s_x(E) \in \mathcal{A}_Y$ and the function $y \mapsto f(x,y)$ is measurable w.r.t. $\mathcal{A}_Y$, for a.e. $x \in X$

   (b) $t_y(E) \in \mathcal{A}_X$ and the function $x \mapsto f(x,y)$ is measurable w.r.t. $\mathcal{A}_X$, for a.e. $y \in Y$

   Notice that we use the notations in Bass's textbook, so $s_x(E)$ and $t_y(E)$ are the x-section and y-section of E correspondingly. Similarly, the functions $y \mapsto f(x,y)$ and $x \mapsto f(x,y)$ can be denoted by $S_x f(y)$ and $T_y f(x)$ correspondingly.

3. Consider the settings of the previous question, with the addition of two σ-finite measures $\mu : \mathcal{A}_X \to [0,\infty]$ and $\nu : \mathcal{A}_Y \to [0,\infty]$. Show that for any $E \in \mathcal{A}_X \otimes \mathcal{A}_Y$:

   (a) The functions $y \to \mu(t_y(E))$ and $x \to \nu(s_x(E))$ are measurable w.r.t. the corresponding σ-algebras $\mathcal{A}_X$ and $\mathcal{A}_Y$.

   (b) $\int\int \chi_E(x,y)d\mu(x)d\nu(y) = \int\int \chi_E(x,y)d\nu(y)d\mu(x)$, where the iterated integration is interpreted as shown in class (and in the textbook).

4. Consider the interval $[0,1]$, the Borel σ-algebra $\mathcal{B}$ on it, the Lebesgue measure m, and the counting measure μ, where both measures are taken on $([0,1], \mathcal{B})$. Show that the "diagonal" set $D = \{(x,x) : x \in [0,1]\} \subseteq [0,1]^2$ is measurable w.r.t. the product σ-algebra $\mathcal{B} \otimes \mathcal{B}$, but

$$\int_0^1 \int_0^1 \chi_D \mu(dy)m(dx) \neq \int_0^1 \int_0^1 \chi_D m(dx)\mu(dy)\,.$$

   Explain why does this example not contradict the Fubini-Tonelli theorem.

5. Let μ be a probability measure on $\mathbb{R}$ (i.e., $\mu(\mathbb{R}) = 1$) and let $h_\mu : \mathbb{R} \to \mathbb{R}$ be defined as $h_\mu(x) = \mu((-\infty, x])$. Given an arbitrary $\varepsilon > 0$, compute the Lebesgue integral $\int \frac{h_\mu(x+\varepsilon) - h_\mu(x-\varepsilon)}{\varepsilon} dx$.

6. Prove that if $(X, \mathcal{A}_X, \mu)$ and $(Y, \mathcal{A}_Y, \nu)$ are σ-finite measure spaces, and λ is some measure on $\mathcal{A}_X \otimes \mathcal{A}_Y$, such that $\lambda(A \times B) = \mu(A)\nu(B)$ for every $A \in \mathcal{A}_X$ and $B \in \mathcal{A}_Y$, then $\lambda = \mu \times \nu$ on $\mathcal{A}_X \otimes \mathcal{A}_Y$.

7. Prove that $\int_{-\infty}^{\infty} |f(x)|dx = \int_0^\infty m(\{x : |f(x)| \geq t\})dt$, where m is, as usual, the Lebesgue measure.

8. Consider two sequences $a_1, a_2 \ldots \in \mathbb{R}$ and $b_1, b_2 \ldots \in \mathbb{R}$, s.t. $\sum_{i=1}^\infty |a_i| < \infty$. Show that $\sum_{i=1}^\infty a_i(|x - b_i|)^{-1/2}$ converges absolutely for almost every $x \in \mathbb{R}$.