Exercise VI - mandatory

Math 320a/520a - Fall Semester 2017

Due Tuesday, 10/17/2017, 2:30 PM

1. Show that for any Lebesgue integrable function $f : \mathbb{R} \to \mathbb{R}$ and nonzero $a \in \mathbb{R}$ the following "change of variable" formulas hold:

   (a) $\int_{\mathbb{R}} f(x+a)dx = \int_{\mathbb{R}} f(x)dx$

   (b) $\int_{\mathbb{R}} f(ax)dx = |a|^{-1} \int_{\mathbb{R}} f(x)dx$

2. Consider a bounded Lebesgue measurable function $g : [0,1] \to \mathbb{R}$. Show that if $\int_0^1 f(x)g(x)dx = 0$ for every continuous $f : [0,1] \to \mathbb{R}$ with $\int_0^1 f dx = 0$ then g is a constant almost everywhere.

3. Prove that a Borel measurable function $f : [-\pi, \pi] \to [-1, 1]$ is *Riemann* integrable if and only if it is continuous almost everywhere (i.e., its set of discontinuities has zero Lebesgue measure), and that in this case the Lebesgue integral agrees with the Riemann one.

4. Consider a measure space $(X, \mathcal{A}, \mu)$. Show that if $f_n \to f$ in measure and $g_n \to g$ in measure then:

   (a) $f_n + g_n \to f + g$ in measure;

   (b) $f_n g_n \to fg$ in measure when μ is a finite measure;

   (c) if $\mu(X) = \infty$ then $f_n g_n$ might not converge to fg in measure.

5. A sequence of functions $f_1, f_2, \ldots : X \to \mathbb{R}$ is said to be *Cauchy in measure* when for every $\varepsilon, a > 0$ there exists some big enough N such that $\mu(\{x : f_n(x) - f_m(x) > a\}) < \varepsilon$ for every $m, n \geq N$. Prove that every sequence that is Cauchy in measure also converges in measure.

6. Consider the space of measurable complex-valued functions over a finite measure space $(X, \mathcal{A}, \mu)$, and suppose we want to compare functions in this space using the following relation:

   $$d(x,y) = \int \frac{|f-g|}{1+|f-g|}$$

   Show that this relation is in fact a distance metric between functions in this space, except that $d(f,g) = 0$ implies $f = g$ a.e. rather than pointwise equality everywhere. Furthermore, prove that $f_n \to f$ in measure if and only if $d(f_n, f) \to 0$.

7. Suppose $(X, \mathcal{A}, \mu)$ is a measure space and X is a countable set. Prove that if $f_1, f_2, \ldots$ is a sequence of measurable functions converging to f in measure, then $f_n \to f$ a.e.

8. Give examples of the following sequences (and measure spaces), and carefully justify them:

   (a) $f_n \to f$ a.e. but does not converge in measure.

   (b) $f_n \to f$ in measure, where the measure is finite, but it does not converge L^2.

   (c) $f_n \to f$ in L^p (for some $p \geq 1$), but it does not converge a.e.