Exercise IV - mandatory

Math 320a/520a - Fall Semester 2017

Due Tuesday, 10/03/2017, 2:30 PM

1. State and prove the **Carathéodory Extension Theorem**. *Notice: this is* **not** *the same as the Carathéodory Theorem, which was shown and proved in class.*

2. Let A be a Lebesgue measurable subset of $\mathbb{R}$ with Lebesgue measure $0 < m(A) < \infty$. Show that:

   (a) There exists a subset of A that is non-measurable (with respect to the Lebesgue σ-algebra).

   (b) For any $\varepsilon > 0$ there exist an open set G and a closed set F such that $F \subseteq A \subseteq G$ and $m(G \setminus F) < \varepsilon$.

   (c) There is a monotonically decreasing series of open sets that "almost converges" to A, i.e., there exist open sets $E_1, E_2, \ldots$ such that $E_i \downarrow E$, $A \subseteq E$ and $m(E \setminus A) = 0$.

3. Prove the **Steinhaus Theorem**, which states that if $A \in \mathcal{B}_{\mathbb{R}}$ is a Borel set with Lebesgue measure $m(A) > 0$ and $B = \{x - y : x, y \in A\}$ then B contains a nonempty open interval centered at the origin.

4. As mentioned in class, the measure space $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, m)$ defined with the Borel σ-algbra and the Lebesgue measure, is not complete.

   (a) Prove that $\mathcal{B}_{\mathbb{R}}$ is a proper subset of the Lebesgue σ-algebra (i.e., every Borel set is Lebesgue measurable, but not every Lebesgue measurable set is a Borel set).

   (b) Show that the Lebesgue σ-algebra is the completion of the Borel σ-algebra with respect to the Lebesgue measure.

5. Show that for every Lebesgue measurable $f : \mathbb{R} \to \mathbb{R}$ there exists a Borel measurable $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ a.e.

6. Show that if m is a Lebesgue-Stieltjes measure corresponding to a right continuous increasing function α, then $m(\{x\}) = \alpha(x) - \lim_{y \to x^-} \alpha(y)$ for every $x \in \mathbb{Q}$.

7. Consider the measurable space $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ and construct a *finite* measure space $(X, \mathcal{A}, \mu)$ and a function $f : X \to \mathbb{N}$ such that f is not $(\mathcal{A}, \mathcal{P}(\mathbb{N}))$-measurable.

8. Give an example of a (possibly uncountable) collection $\{f_\alpha\}_{\alpha \in A}$ of Borel measurable non-negative functions whose supremum $g(x) = \sup_{\alpha \in A} f_\alpha(x)$ is finite but not Borel measurable. Can you find a countable collection that satisfies this condition?