Exercise II - mandatory

Math 320a/520a - Fall Semester 2017

Due Tuesday, 09/19/2017, 2:30 PM

1. Given a σ-algebra[1] $\mathcal{A}$ on a set X, show that:
   (a) $\{\emptyset, X\} \subseteq \mathcal{A}$
   (b) $A \setminus B \in \mathcal{A}$ for any $A, B \in \mathcal{A}$
   (c) $\bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$ for any countable series of sets $A_1, A_2, \ldots \in \mathcal{A}$

   Notice: some definitions of a σ-algebra use property (a) (or just $\emptyset \in \mathcal{A}$) as a requirement instead of the nonempty one ($\mathcal{A} \neq \emptyset$). This exercise shows that both versions are equivalent.

2. Given a positive integer n, let $\mathcal{A}_n \subseteq \mathcal{P}(\mathbb{Z})$ be defined as $\mathcal{A}_n = \Big\{A : \text{either } A \subseteq \{1, \ldots, n\}$ or $A^c \subseteq \{1, \ldots, n\}\Big\}$, where complements are taken relative to the set $\mathbb{Z}$ of all integers. For which values of n is $\mathcal{A}_n$ a σ-algebra on $\mathbb{Z}$? Prove your answer.

3. Given two nonempty disjoint subsets $A, B \subsetneq X$, what is the size (i.e., number of elements) of the smallest σ-algebra $\sigma(\{A, B\})$ containing them? Explain how you calculated your answer.

4. Suppose $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \mathcal{A}_3 \subseteq \ldots \subseteq \mathcal{P}(\mathbb{Z})$ are σ-algebras on the integers. Is $\bigcup_{i=1}^{\infty} \mathcal{A}_i$ necessarily a σ-algebra? If yes, prove it; if not, give a counterexample.

5. For any function $f : X \to Y$, we define its inverse $f^{-1} : \mathcal{P}(Y) \to \mathcal{P}(X)$ on subsets of Y as $f^{-1}(A) = \{x \in X : f(x) \in A\}$ for any $A \subseteq Y$. Prove that if $\mathcal{A}_Y$ is a σ-algebra on Y, then $\mathcal{A}_X = \{f^{-1}(A) : A \in \mathcal{A}_Y\}$ is a σ-algebra over X.

6. Show that the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ over the real numbers is equivalently generated by each of following collections of sets:
   (a) $\{[a, b] : a, b \in \mathbb{R}\}$
   (b) $\{(a, \infty) : a \in \mathbb{R}\}$
   (c) $\{(-\infty, b] : b \in \mathbb{R}\}$

[1] Using the definition shown in class, i.e., a σ-algebra on X is a nonempty collection $\mathcal{A} \subseteq \mathcal{P}(X)$ that is closed under complements and countable unions.