Exercise X

Math 305b/ ENAS 514b - Spring Semester 2017

Due Wednesday, 04/26/2015, 1:00 PM

The following problems are based on the definitions in Beals 14G, and provide a justification for the FFT algorithm shown in class:

1. Let $e_{n,N}(t) = e^{i2\pi nt/N}$, $n = 0, \ldots, N-1$, $t \in \mathbb{Z}$. Show that (a) these vectors are N periodic (i.e., $e_{n,N}(t+N) = e_{n,N}(t)$), (b) they form an orthonormal basis to the space of N periodic functions based on the inner products defined in class (and in Beals 14G, eq. 28), and (c) negative "frequencies"[1] are considered in this case by $e_{-k,N} = e_{N-k,N}$ for any $k = 1, \ldots, N-1$.

2. Let f be an N periodic function with $N = 2M$ for some $M \in \mathbb{N}$, and let the functions f_+ and f_- be as defined in class (and in Beals 14G, eq. 33). Show that (a) these functions are M periodic, and (b) their Fourier coefficients $\widehat{f_+}(k)$ and $\widehat{f_-}(k)$ (computed using a period of M) give the Fourier coefficients $\widehat{f}(2k)$ and $\widehat{f}(2k+1)$ (correspondingly) of f (computed using a period N) for any $k = 0, \ldots, M-1$.

The following problems refer to the general Fourier transform of functions on $\mathbb{R}$, as defined in Beals 14H (eq. 46). The concerned functions are assumed to be sufficiently "nice" (i.e., satisfy appropriate conditions) to justify the manipulations in the problem:

3. Show that if f is integrable then its Fourier transform $\widehat{f}$ is continuous.

4. Show that the derivative $f' = \frac{df}{dt}$ has the Fourier transform $\widehat{f'}(\omega) = i\omega\widehat{f}(\omega)$.

5. Show that if $h(t) = tf(t)$ then its Fourier transform is $\widehat{h}(\omega) = i\frac{d\widehat{f}(\omega)}{d\omega}$.

6. Let $f_a = f(t-a)$ for some $a \in \mathbb{R}$. Compute an expression for $\widehat{f_a}(\omega)$ in terms of $\widehat{f}(\omega)$ (simplify as much as possible) and justify your answer.

7. Let $f_\rho = f(\rho t)$ for some $\rho > 0$. Compute an expression for $\widehat{f_\rho}(\omega)$ in terms of $\widehat{f}(\omega)$ (simplify as much as possible) and justify your answer.

8. Show that the Fourier transform of a Gaussian $g(t) = \frac{1}{\sqrt{2\pi}}e^{-t^2/2}$ is given by $\widehat{g}(\omega) = e^{-\omega^2/2}$.

[1] Obtained by simply allowing negative indices n in the definitions of $e_{n,N}$