Exercise V

Math 305b/ ENAS 514b - Spring Semester 2017

Due Monday, 02/27/2015, 1:00 PM

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function.
   (a) Show that if f vanishes (i.e., equals zero) outside some finite measure set, then it is integrable.
   (b) Is this still true when $m(\{f \neq 0\}) = \infty$? (prove or give a counter example)
2. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Show that if f is bounded then $\int f^2 < \infty$.
3. Find a bounded measurable nonnegative function $g : \mathbb{R} \to \mathbb{R}$ such that $\int g^2 < \infty$ but $\int g = \infty$.
4. Show that if $f : \mathbb{R} \to \mathbb{R}$ is a measurable function then $\int f^+ = \lim_{n\to\infty} \int (f^+ \wedge n)$ and $\int f^- = \lim_{n\to\infty} \int (f^- \wedge n)$. Use this result to show that if f is integrable and

$$f_n(x) = \begin{cases} n & f(x) > n \\ f(x) & -n \leq f(x) \leq n \\ -n & f(x) < -n \end{cases}$$

   for $n = 1, 2, \ldots$, then f_n are all integrable and $\int f = \lim_{n\to\infty} \int f_n$.
5. Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable nonnegative function. Prove that $m(\{x : f(x) > a\}) \leq a^{-1} \int f$ for any $a > 0$.
6. Show that there exists a measurable function $f : \mathbb{R} \to \mathbb{R}$ that satisfies (a) $f(x) = 0$ for every $x \notin [0, 1]$, and (b) for every $L \in \mathbb{R}$ there exists a sequence of ISF's g_n, $n = 1, 2, \ldots$, such that $\lim_{n\to\infty} \int g_n = L$ and $\lim_{n\to\infty} g_n(x) = f(x)$ for every $x \in \mathbb{R}$.
7. Find a sequence pf nonnegative measureable functions f_n, $n = 1, 2, \ldots$, with the following properties: (a) $f_n(x) = 0$ for every $x \notin [0, 1]$, (b) $\lim_{n\to\infty} f_n(x) = 0$ for every $x \in [0, 1]$, and (c) $\lim_{n\to\infty} \int f_n = \infty$.
8. Show that if $f : \mathbb{R} \to \mathbb{R}$ is a nonnegative, bounded, integrable function that vanishes outside $[0, 1]$, its integral satisifies $\int f = \inf\{\int g : f \leq g, g \text{ is an ISF}\}$.