Exercise IV

Math 305b/ ENAS 514b - Spring Semester 2017

Due Monday, 02/20/2015, 1:00 PM

1. Prove the following results for $f : \mathbb{R} \to \mathbb{R}$:
   (a) The sets $\{x : f(x) > r\}$, $r \in \mathbb{Q}$, are all measurable if and only if f is measurable;
   (b) If f is a monotone function then f is measurable.
2. Prove that if $f_1, f_2, \ldots : \mathbb{R} \to \mathbb{R}$ are measureable functions, then the set $\{x : \lim_{n\to\infty} f_n(x) \text{ exists}\}$ is measurable. *Notice:* the limit here does not have to be finite.
3. Let $f_1, f_2, \ldots : \mathbb{R} \to \mathbb{R}$ be a sequence of measurable functions that converges to a function $f : \mathbb{R} \to \mathbb{R}$ at every point $x \in A$ for a measurable A with finite measure. Prove that for any $\varepsilon > 0$ there exist a set $B \subseteq A$ such that $m(B) < \varepsilon$ and $\{f_n\}$ converges *uniformly* to f on $A \setminus B$. This result is called *Egorov's Theorem.*
4. Let the binary expansion of $x \in \mathbb{R}$ be $x = \sum_{n=1}^{\infty} \frac{\varepsilon_n}{2^n}$, $\varepsilon_n \in \{0, 1\}$, and consider the following function, which was defined and examined in class and in Beals 11B (page 148):

$$g(x) = \begin{cases} \sum_{n=1}^{\infty} \frac{2\varepsilon_n}{3^n} & x \in (0,1]\,; \\ 0 & \text{otherwise}\,. \end{cases}$$

   Show that the function g is continuous at every point *except* for $\frac{p}{2^n}$, $p, n \in \mathbb{N}$.
5. Give an example of measurable functions $f, g : \mathbb{R} \to \mathbb{R}$ such that their composition $f \circ g$ is not measurable.
6. Show that if f is an ISF (integrable simple functions) and $c \in \mathbb{R}$, then:
   (a) The following functions are also ISF: $c\,f$, $|f|$, and the translation $f_c(x) = f(x - c)$;
   (b) The integrals of these functions satisfy $\int (c\,f) = c \int f$, $\left|\int f\right| \le \int |f|$, and $\int f_c = \int f$.
7. Let f, g be ISF , show that
   (a) Their sum $f + g$ is also an ISF, and $\int (f + g) = \int f + \int g$;
   (b) If $f(x) \le g(x)$ for every $x \in \mathbb{R}$ then $\int f \le \int g$.
8. Prove that f is an ISF if and only if (a) f is measurable, (b) f takes finitely many values, and (c) $m(f^{-1}(\mathbb{R} \setminus \{0\})) < \infty$.