Exercise III

Math 305b/ ENAS 514b - Spring Semester 2017

Due Monday, 02/13/2015, 1:00 PM

1. Let A be the nonmeasurable set "constructed" in class (and in Beals 10E). Note that, depending on exact choices, there can be many possiblities for A.
   (a) Show that for any $\varepsilon > 0$, one can construct A such that $m^*(A) < \varepsilon$.
   (b) Can A constructed to have $m^*(A) = 0$? Explain your answer.
2. Let $A \subseteq \mathbb{R}$ be a measurable set. Show that for any $\alpha \in \mathbb{R}$, the set $\alpha A \triangleq \{\alpha x : x \in A\}$ is measurable and its measre is $m(\alpha A) = |\alpha| m(A)$.
3. Find an example of measurable subsets $A_1, A_2, A_3, \ldots$ of $[0,1]$ such that $m(A_i) > 0$, $m(A_i \Delta A_j) > 0$ (where Δ denotes the symmetric set difference), and $m(A_i \cap A_j) = m(A_i)m(A_j)$ for every $i, j \in \mathbb{N}$, $i \neq j$.
4. Prove that any measurable set $A \subseteq \mathbb{R}$ with $m(A) > 0$ has a non-measurable subset.
5. Show that for any measurable (possibly uncountable) set $A \subseteq \mathbb{R}$ the set $B = \bigcup_{x \in A} [x - 1, x + 1]$ is also measurable.
6. Let $A_1, A_2, \ldots \subseteq \mathbb{R}$ be countably many pairwise disjoint subsets of $\mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be defined as:

$$f(x) = \begin{cases} n & \text{if } x \in A_n \\ 0 & \text{otherwise} \end{cases}$$

   Prove that f is measurable if and only if A_n is a measurable set for every n.
7. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is continuous and $g : \mathbb{R} \to \mathbb{R}$ is measurable then their composition $(f \circ g)(x) = f(g(x))$ is measurable.
8. Let $f, g : \mathbb{R} \to \mathbb{R}$ be two measurable functions. Prove the following:
   (a) If A is an open subset of $\mathbb{R}^2$ then $\{x \in \mathbb{R} : \big(f(x), g(x)\big) \in A\}$ is a masurable set.
   (b) If $H : \mathbb{R}^2 \to \mathbb{R}$ is continuous then $h(x) = H\big(f(x), g(x)\big)$ is a measurable function.
   (c) Using the previous part, prove that $f + g$ and fg are both measurable functions.